Algebra Prelim part A

January 7, 2014

Directions: You have 90 minutes. Answer two questions; specify clearly which problems you want graded.

A1. Let G be the group of two-by-two matrices with entries in $\mathbf{Z}/p\mathbf{Z}$ and non-zero determinant, where p is a prime number.

(a) Show that the group H of upper-triangular matrices with ones along the diagonal is a p-Sylow subgroup of G.

(b) Show that the normalizer of H equals the group of upper-triangular matrices with non-zero determinant.

(c) Show that the number of p-Sylow subgroups of G is $p + 1$.

A2. Suppose G is a group acting faithfully on a set X and N is a nontrivial normal subgroup.

(a) Show that for any $g \in G$ and any orbit Y of N, $g(Y)$ is an orbit of N.

(b) Show that if G acts 2-transitively on X then N acts transitively.

(c) Give an example of the situation in (b), with N acting simply transitively.

A3. Let T be a linear transformation of a vector space V over $\mathbf{Q}$, of order a prime p. State and prove the relationship between the rational canonical form of T and the rational canonical form of the map it induces on $V \otimes V$.