PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

August 20, 2014, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let X and Y be Banach spaces, $T \in B(X,Y)$, and T be bounded below.

(**a**) Show that T is a injective.

(**b**) Show that the range of T, $R(T)$, is closed in Y.

(**c**) Give a simple example of T that is bounded below but *not* surjective.

(**d**) Define $\tilde{T} : X \to R(T)$ by $\tilde{T}x = Tx$ for all $x \in X$. Show that $\tilde{T}$ is a bijective, bounded linear map.

2. Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $a : \Omega \to \mathbb{R}$ define

$$(Tu)(x) = a(x)\, u(x) \quad \forall x \in \Omega.$$

Assume $Tu \in L^q(\Omega)$ for every $u \in L^p(\Omega)$ for some $1 \le q \le p \le \infty$.

(**a**) Show that the map $T : L^p(\Omega) \to L^q(\Omega)$ is bounded. [Hint: Consider uniform boundedness of a sequence of approximating operators.]

(**b**) Show that $a \in L^r(\Omega)$, where $r = pq/(p-q)$ if $p < \infty$ and $r = q$ if $p = \infty$.

3. Let X and Y be Banach spaces, let $A : X \to Y$ be bounded, linear and surjective, let $B : X \to Y$ be bounded and linear, and let $\alpha = ||A - B||$.

(**a**) Show that there exists $\sigma > 0$ such that $\bar{B}^Y_r \subset A\bar{B}^X_{r/\sigma}$ for all $r > 0$, where $\bar{B}^X_r$ and $\bar{B}^Y_r$ are *closed* balls of radius r about the origin in X and Y, respectively.

(**b**) For given $f \in Y$, let $y_n \in Y$ and $x_n \in X$ be sequences such that

$$y_0 = f, \quad Ax_n = y_n, \quad \text{and} \quad y_{n+1} = y_n - Bx_n \quad \text{for } n \ge 0.$$

Show that the required x_n can be chosen such that

$$||y_n|| \le (\alpha/\sigma)^n ||f|| \quad \text{and} \quad ||x_n|| \le \sigma^{-1}(\alpha/\sigma)^n ||f|| \quad \text{for } n \ge 0.$$

[Hint: Use induction.]

(**c**) If α is sufficiently small, show that $\displaystyle\sum_{n=0}^{\infty} x_n$ converges and $B\Big(\displaystyle\sum_{n=0}^{\infty} x_n\Big) = f$, and conclude that B must also be surjective.

PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

August 20, 2014, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

4. Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space and $\hat{f}$ denote the Fourier transform of $f \in \mathcal{S}$.

(**a**) Prove that for $f, \phi \in \mathcal{S}$,

$$\lim_{\epsilon \to 0^+} \int f(x)\, \epsilon^{-d} \hat{\phi}(x/\epsilon)\, dx = f(0) \int \hat{\phi}(x)\, dx$$

(**b**) Prove that for $f \in \mathcal{S}$,

$$f(x) = (2\pi)^{-d/2} \int \hat{f}(\xi)\, e^{i\, x \cdot \xi}\, dx.$$

5. Let H and W be real Hilbert spaces and let $V \subset H$ be a linear subspace. Let $A : H \to H$ and $B : V \to W$ be bounded linear operators, where we give V the norm $\|v\|_V = \|v\|_H + \|Bv\|_W$. For any $f \in V$ and $0 \le \delta < 1$, consider the problem: Find $(u, p) \in V \times W$ such that

$$\begin{aligned} &\langle Au, v\rangle_H - \langle B^* p, v\rangle_H + \langle Bu, w\rangle_W + \langle p, w\rangle_W + \delta\langle Bu, Bv\rangle_W + \delta\langle p, Bv\rangle_W \\ &\quad = \langle f, v\rangle_H \quad \forall (v, w) \in V \times W. \end{aligned}$$

Assume that A is coercive on V (i.e., there is $\alpha > 0$ such that $\alpha\|v\|_V^2 \le \langle Av, v\rangle_V$).

(**a**) Assuming there is a solution, find a bound on the norm of the solution (u, p).

(**b**) Show that there is a unique solution for any $\delta \in (0, 1)$.

(**c**) Show that there is a unique solution for $\delta = 0$. [Hint: Replace w by $w - \delta Bv$.]

6. Let X and Y be normed vector spaces, and let $[a, b]$ and (a, b) denote closed and open line segments between two given points $a, b \in X$.

(**a**) Let $f : X \to Y$ be a function which is continuous on the segment $[a, b]$ and differentiable on the segment (a, b), and let $A \in B(X, Y)$ be given. Show that

$$||f(b) - f(a) - A(b - a)||_Y \le M||b - a||_X, \quad \text{where} \quad M = \sup_{x \in (a,b)} ||Df(x) - A||_{B(X,Y)}.$$

(**b**) Let $g : X \to Y$ be a function which is continuous in X and differentiable in $X \setminus \{a\}$. Show that, if $L := \lim_{x \to a} Dg(x)$ exists, then g is differentiable at a and $Dg(a) = L$.

(**c**) Consider $g : X \to \mathbb{R}$ where $g(x) = ||x||_X$. Show that g cannot be differentiable at $x = 0$.