THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part II

August 22nd, 2014

Problem 2.1. Consider a process M which is adapted to the filtration $(\mathcal{F}_t)_{0\leq t} < \infty$, jointly measurable, and satisfies, for each $t \geq 0$ that $\mathbb{E}[|M_t|] < \infty$, and $\mathbb{E}[\int_0^t |M_s|] < \infty$. Show that M is a martingale if and only if

$$\forall\ 0 \leq s < t, \qquad \mathbb{E}\left[\frac{\int_s^t M_u du}{t-s} | \mathcal{F}_s\right] = M_s.$$

Problem 2.2. Let $(W_t)_{0\leq t\leq 1}$ a Brownian motion (defined only up to time one). Show that the two dimensional random vector

$$\left(W_1, \int_0^1 sgn(W_s) dW_s\right)$$

has the following properties

(1) both marginals are normal $N(0,1)$
(2) however, it is NOT a joint normal random vector

Problem 2.3. (the Stratonovich integral and the chain rule) For two continuous semi-martingales X and Y (on the same space and filtration), we define the Stratonovic integral

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_s,$$

where $\int_0^t X_s dY_s$ represents the Itô integral. Show that, if f is a C^3 function, and X is a continuous semi-martingale, then, we have the chain rule

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s,$$

(or, in differential notation, that $df(X_t) = f'(X_t) \circ dX_t$.) Do we need f to be C^3?

Note: please note that the Stratonovich integral matches the chain rule we know from deterministic calculus, at the formal level. However, this integral does not match the intuition we have for the integral as cumulative gains/losses from a gambling/investment strategy).