Algebra Prelim Part A January 2016

Do two of the three questions below. Please indicate which questions you want graded.

1. Let R be an integral domain such that every prime ideal of R is principal.

   a) Consider the set of ideals of R which are not principal. Prove that if this set is non-empty, then it contains an element I which is maximal under inclusion.

   b) Prove that R is a principal ideal domain. (Hint: Show that I is principal.)

2. Let R be a PID, π an irreducible element of R and consider the subset M of R^2 of pairs (x, y) with π dividing y and π^2 dividing $y - x\pi$. Show that M is a submodule of R^2 of rank 2 and find a basis v_1, v_2 of R^2 and $a_1, a_2 \in R$ with a_1 dividing a_2 such that a_1v_1, a_2v_2 is a basis of M.

3. Let H be a subgroup of a finite group G. Define $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove that $N_G(H)$ is a subgroup of G and that H is normal subgroup of $N_G(H)$. Let $G = SL_2(k)$ the group of 2×2 matrices of determinant one over a field k. Let H be the subgroup of G consisting of upper triangular matrices with ones along the diagonal. Compute $N_G(H)$.