PRELIMINARY EXAMINATION IN ANALYSIS
PART II - COMPLEX ANALYSIS
JANUARY 11, 2016

Please try to solve 4 of the following 5 problems.

(1) Suppose that f is a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and that f is injective on some annulus $\{z \in \mathbb{C} : r < |z| < 1\}$. Show that f is injective on $\mathbb{D}$.

(2) Find all entire functions f that satisfy $f(\sqrt{n}) = n^2$ for every positive integer n, and $|f(z)| \leq e^{3|z|}$ for every complex number z.

(3) Let $f_1, f_2, f_3, \ldots$ be analytic functions, defined on some domain $\Omega \subset \mathbb{C}$, and assume that $f_n \to f$ pointwise on Ω. If none of the functions f_n takes on any positive real values, show that f is analytic on Ω.

(4) Show that the equation $\sin(f(z)) = z$ has a solution f that is analytic in the region $\Omega = \{z \in \mathbb{C} : |z| < 1 \text{ or } \operatorname{Im}(z) \neq 0\}$.

(5) Consider the set S of all analytic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that are one-to-one and satisfy $f(0) = 0$ and $f'(0) = 1$. Show that if $f \in S$ then there exists an odd function $g \in S$ such that $g(z)^2 = f(z^2)$, for all $z \in \mathbb{D}$.