PRELIMINARY EXAMINATION IN ALGEBRA, PART I
JANUARY 2019

1. Questions

(1)
   (a) Classify all finitely generated abelian groups such that the group of automorphisms is finite.
   (b) What is $\mathrm{Aut}(\mathbb{Z} \times C_p \times C_q)$, for p and q relatively prime?

(2)
   (a) Prove that any group of order p^2 is abelian.
   (b) Classify the groups of order p^2.
   (c) Classify the non-abelian groups of order 8.

(3) Suppose that R is a commutative ring with an ideal $I \subset R$ such that $I^2 = I$. Show that $R \cong I \oplus \widetilde{R}$, where $\widetilde{R}$ is another ideal.

(4) Let G be a finite group and H a subgroup of finite index. Suppose that for any $x, y \in G \setminus H$ we have $xH \cap Hy \neq \emptyset$. What can you say about the order of G? (Hint: let G act on $G/H \times G/H$.)

(5) Let R be a ring.
   (a) Show that a finitely presented R-module M is finitely generated. Recall that a module M is finitely presented if there exists an exact sequence
   $$R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0.$$
   (b) Show that given a short exact sequence of R-modules
   $$0 \longrightarrow P \longrightarrow Q \longrightarrow M \longrightarrow 0$$
   such that M is finitely presented and Q is finitely generated, then P is finitely generated.
   (c) Assume that R is Noetherian. Show that a finitely generated R-module M is finitely presented.
   (d) Give an example of a ring R and a module M which is finitely generated but not finitely presented.