PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

January 18, 2019, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space.

(**a**) Prove that a linear operator $T : H \to H$ is self adjoint if and only if $\langle Tx, x \rangle$ is real for all $x \in H$.

(**b**) State the Spectral Theorem for a compact, self-adjoint operator.

(**c**) Let T be a compact, self-adjoint operator. Prove that there are positive operators P and N such that $T = P - N$ and $PN = 0$.

2. Let X and Y be separable and reflexive Banach spaces.

(**a**) Show that if $x_n \overset{w}{\rightharpoonup} x$ in X, then $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$.

(**b**) State the Banach-Alaoglu Theorem.

(**c**) Suppose that $D \subset Y$ is a dense subset. Let $T : X \to Y$ be a bounded linear map that is bounded below. Suppose further that, given any $g \in D$, $Tx = g$ is uniquely solvable for $x \in X$. Prove that there is a unique solution $x \in X$ to $Tx = f$ for any $f \in Y$.

3. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space of real or complex functions defined on $\mathbb{R}^d$. A *reproducing kernel function* for H is a function $K(\cdot, \cdot)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies

i. $K(\cdot, y) \in H$ for all $y \in \mathbb{R}^d$ and

ii. $f(y) = \langle f, K(\cdot, y) \rangle$ for all $f \in H$ and $y \in \mathbb{R}^d$.

(**a**) Prove that H has a reproducing kernel function if and only if for each $y \in \mathbb{R}^d$, the linear functional L_y defined by $L_y(f) = f(y)$ is bounded.

(**b**) If K exists, show that K is unique and that $K(x, y) = \overline{K(y, x)}$.

(**c**) If K exists, $L \in H^*$, and $z(y) = \overline{LK(\cdot, y)}$, show that $z \in H$ and $\|L\|^2 = Lz$.

PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

January 18, 2019, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

1. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on $\mathbb{R}$. Under what conditions on the complex sequence $\{a_k\}_{k=1}^{\infty}$ is $\displaystyle\sum_{k=1}^{\infty} a_k\,\delta_k \in \mathcal{S}'(\mathbb{R})$? Here, δ_k is the point mass centered at $x = k$.

2. Show the following two statements about Sobolev spaces, where $\Omega \subset \mathbb{R}^d$ is a domain.

(**a**) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le p \le d$ and $q > dp/(d-p)$. [Hint: Show a counterexample with the function $f(x) = |x|^\alpha$ by choosing an appropriate domain Ω and exponent α.]

(**b**) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow C^0_B(\Omega)$ for $1 \le p < d$. Note that in the previous case, f is not bounded. What can you say about which (negative) Sobolev spaces the Dirac mass lies in?

3. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and $\{u_k\} \subset H^{2+\varepsilon}(\Omega)$ is a bounded sequence, where $\varepsilon > 0$.

(**a**) Show that there is $u \in H^2(\Omega)$ such that, for a subsequence, $\{u_{k_j}\}_{k=1}^{\infty} \to u$ in $H^2(\Omega)$.

(**b**) Find all q and $s \ge 0$ such that, for a subsequence, $\{u_{k_j}\} \to u$ in $W^{s,q}(\Omega)$.