Differential Topology Prelim

Wednesday, January 16, 2019, 2:40 – 4:15 PM

Instructions: Do all three problems. All manifolds are assumed to be smooth. If you have any questions, ask!

1. Let X^n be an n-manifold, Y^n an m-manifold, and $f : X^n \to Y^m$ a smooth map.

a) Show that if p is a regular value of f, then $f^{-1}(p)$ is a codimension-m submanifold of X. You may use without proof the inverse function theorem and its corollaries.

b) If $f^{-1}(p)$ is a codimension-m submanifold of X, does that imply that p is a regular value? Either prove it is or give a counterexample.

c) Let $g : X^n \to \mathbb{R}^5$ be a smooth map. Show that there exists a 2-plane $P \subset \mathbb{R}^5$ such that $g^{-1}(P)$ is a codimension-3 submanifold of X.

2a) Show that $\mathbb{CP}^1$ is a smooth 2-manifold.

b) Writing $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, let $f_0 : \mathbb{CP}^1 \to \mathbb{CP}^1$ be defined by $f_0(z) = \begin{cases} z^3 & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$. Show that f_0 is a smooth map. (This is trivial everywhere except around $z = \infty$, so you only need to write down your argument for what happens around $z = \infty$.)

c) Homotope f_0 to a map f_1 that has ∞ as a regular value. Then compute the degree of f_0 by counting (with sign) the points of $f_1^{-1}(\infty)$.

3a) On $X = \mathbb{R}^3 - \{0\}$, consider the differential form

$$\alpha = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{4\pi(x^2 + y^2 + z^2)^{3/2}}.$$

Show that α is closed but not exact.

b) Let Y and Z be isotopic embeddings of a genus-g oriented surface M_g in X. That is, there is a smooth map $I : [0,1] \times M_g \to X$ such that i_0 is the embedding of Y, i_1 is the embedding of Z, and each i_t is an embedding of M_g in X. Give Y and Z orientations inherited from M_g. Show that $\int_Y \alpha = \int_Z \alpha$.

c) You can take as a given the fact that Y divides $\mathbb{R}^3$ into two pieces, namely an inside and an outside. Show that $\int_Y \alpha = \pm 1$ if the origin is on the inside, and that $\int_Y \alpha = 0$ if the origin is on the outside.