PRELIMINARY EXAMINATION IN TOPOLOGY: PART 2

August 2021 2 Hours

Work all 3 problems. Explain your work carefully.

1. For each of the following statements give a proof or a counterexample. Justify your counterexamples.
   (a) Let X be a smooth compact manifold, and suppose $f, g\colon X \to X$ are smooth maps whose Lefschetz numbers satisfy $L(f) = L(g)$. Then f is homotopic to g.
   (b) Suppose $f\colon X \to Y$ is a smooth map between smooth manifolds, and $y_0, y_1 \in Y$ are regular values of f. Assume that Y is connected. Then the manifolds $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are diffeomorphic.
   (c) Suppose $f\colon X \to Y$ is a smooth map from one smooth connected manifold to another. Then any two regular values of f are connected by a smooth path of regular values.

2. (a) Construct a 1-form $\omega \in \Omega^1(S^1 \times S^1)$ and a smooth curve $\gamma\colon S^1 \to S^1 \times S^1$ such that

$$\int_{S^1} \gamma^*\omega \neq 0$$

   and $d\omega = 0$.
   (b) Prove that for any closed 1-form $\omega \in \Omega^1(S^2)$ and any smooth curve $\gamma\colon S^1 \to S^2$, we have

$$\int_{S^1} \gamma^*\omega = 0.$$

   (c) Prove that $S^1 \times S^1$ is not diffeomorphic to S^2.

3. (a) Suppose that X is an n dimensional manifold and $f^1, \ldots, f^{n-1}\colon X \to \mathbb{R}$ are smooth functions such that the differentials $df^1_x, \ldots, df^{n-1}_x$ are linearly independent at some point $x \in X$. Prove that there exists a function $f^n\colon X \to \mathbb{R}$ such that $f^1, \ldots, f^n$ is a local coordinate system in a neighborhood of x.
   (b) Suppose $f\colon X \to Y$ is a smooth map from a compact oriented nonempty manifold X to a connected manifold Y. Assume that df_x is invertible for all $x \in X$. Prove that f is surjective.