

ALGEBRA PRELIMINARY EXAM: PART I

PROBLEM 1

- a) Let L be a field, $V := L^n$ the L -vector space of column vectors with n entries, and $R := L[T]$ the ring of polynomials with coefficients in L .
- Explain why an R -module structure on V , restricting to its natural L -module structure, determines and is determined by an $n \times n$ matrix over L .
 - For an $n \times n$ matrix A , let V_A be the corresponding R -module. Show that V_A, V_B are isomorphic if and only if the matrices A and B are conjugate.
- b) Let $L \subset K$ be a field extension. Let A, B be $n \times n$ matrices over L which are conjugate over K . Are they necessarily conjugate over L ? If so, prove it; if not, give a counter-example.

PROBLEM 2

- a) Prove that a finite group is not the union of conjugates of any proper subgroup.
- b) Show by way of example that this fails in general for infinite groups. Equivalently: give an example of a proper subgroup H of a group G such that every element of G is conjugate to an element of H .

PROBLEM 3

Let p be the smallest prime dividing the order of a finite group G , and $H \subset G$ a subgroup of index p . Show that H is normal.