ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let p be a prime.

a) Let $f(x) = x^p - x + 1 \in \mathbb{F}_p(x)$ and α be a root of $f(x)$. Prove that $\mathbb{F}_p(\alpha)/\mathbb{F}_p$ is Galois and determine the cardinality of $\mathbb{F}_p(\alpha)$.
b) Prove that $\mathbb{F}_p(x, y)/\mathbb{F}_p(x^p, y^p)$ is not a simple extension.

Problem 2

Consider $f(x) = x^5 + 20x + 16 \in \mathbb{Q}[x]$.

a) Determine the Galois group of $f(x)$ over $\mathbb{Q}$ (as a subgroup of S_5).
b) Determine whether $f(x) = 0$ is solvable by radicals.

In the solution of this problem you may use without proof the following facts:
i) the discriminant of $f(x)$ is $2^{16} \cdot 5^6$,
ii) a transitive subgroup of S_5 is isomorphic to one and only one of the following groups: $\mathbb{Z}_5$, $\mathrm{F}_{20} :=< \sigma, \tau > / < \sigma^5 - 1, \tau^4 - 1, \sigma\tau - \tau\sigma^2 >$ (the Frobenius group of order 20), D_{10} (the Dihedral group of order 20), A_5, S_5.

Problem 3

Let p be a prime, $n \in \mathbb{N}$, and ζ_n a primitive n-th root of unity.

a) Prove that the Galois group of $x^p - 2$ is isomorphic to the group of matrices
$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_p, a \neq 0 \right\}.$$
b) Prove that $\mathbb{Q}(\sqrt[5]{2})$ is not a subfield of $\mathbb{Q}(\zeta_n)$ for any $n \in \mathbb{N}$.

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