PRELIMINARY EXAMINATION IN ANALYSIS

PART I

AUGUST 2023

Please provide as-complete-as-possible proofs for all of the following 4 problems.

(1) Let $1 < p < \infty$ and $f_n \in L^p([0,1],\lambda)$ $(n = 1,2,\ldots)$ and $f \in L^p([0,1],\lambda)$ (where λ is Lebesgue measure). Let $M > 0$ be a constant and suppose

(a) f_n converges to f pointwise a.e.;

(b) $\|f_n\|_p \leq M$ for all n.

Prove that for every $g \in L^q([0,1],\lambda)$ $(\frac{1}{p} + \frac{1}{q} = 1)$,

$$\lim_{n\to\infty} \int f_n g \, \mathrm{d}\lambda = \int fg \, \mathrm{d}\lambda.$$

Warning: do not assume f_n converges to f in $L^p([0,1],\lambda)$.

(2) Let ν be a finite Borel measure on $[0,1]$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \nu([0,x)).$$

Prove ν is absolutely continuous to Lebesgue measure if and only if f is absolutely continuous.

(3) Given a function $\phi : \mathbb{R} \to [0,\infty)$ and $\epsilon > 0$, define $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ by $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$. Let $f \in \mathrm{L}^\infty(\mathbb{R},\lambda)$ (where λ is Lebesgue measure).

(a) Fix $0 < a < 1$ and let $\psi = (1/2a)1_{[-a,a]}$ (where $1_{[-a,a]}$ is the characteristic function of $[-a,a]$). Explain why the convolution $f * \psi_\epsilon$ converges to f pointwise a.e. as $\epsilon \searrow 0$.

(b) Suppose $\phi : \mathbb{R} \to [0,\infty)$ be a continuous function which is even (so $\phi(x) = \phi(-x)$ for all x), has $\phi(x) = 0$ for all $|x| \geq 1$ and has $\|\phi\|_1 = 1$. Prove $f * \phi_\epsilon \to f$ pointwise a.e. as $\epsilon \searrow 0$.

(4) Let μ_n be a sequence of Borel probability measures on $[0,1]$ which converges to a probability measure μ in the weak* topology. This means: for every continuous function f on $[0,1]$, $\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f \, \mathrm{d}\mu_n$.

Let $C \subset [0,1]$ be closed. Prove $\limsup_{n\to\infty} \mu_n(C) \leq \mu(C)$.