THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Thu, Aug 17, 2023

Problem 1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables taking values in $\{0,1\}$, with $\mathbb{P}[X_n = 1] = p_n \in [0,1]$. Find necessary and sufficient conditions on the sequence $\{p_n\}_{n\in\mathbb{N}}$ such that $\{X_n\}_{n\in\mathbb{N}}$ converges in a) $\mathbb{L}^\infty$, b) $\mathbb{L}^1$, c) a.s., d) probability and e) distribution.

Problem 2. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an iid sequence with $\mathbb{E}\left[\xi_i^4\right] < \infty$. Prove that $\frac{1}{n}S_n \to \mathbb{E}[\xi_1]$, a.s., where $S_n = \sum_{i=1}^n \xi_i$. (*Hint:* Show that $\mathbb{E}\left[\sum_n (S_n/n)^4\right] < \infty$ when $\mathbb{E}[\xi_i] = 0$.)

Problem 3. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a sequence of sub-σ-algebras of $\mathcal{F}$. Show that $X_n \xrightarrow{\mathbb{P}} 0$ if $\mathbb{E}[X_n|\mathcal{F}_n] \xrightarrow{\mathbb{P}} 0$. Does the converse hold? (*Hint:* Identify a function f with the property that $X_n \xrightarrow{\mathbb{P}} 0$ if and only if $\mathbb{E}[f(X_n)] \to 0$.)