ALGEBRA I QUALIFYING EXAM

JANUARY 13TH, 2023

Each problem is worth 10 points. A passing score is 20/30.

Problem 1.

(a) Let G be a finite group of order n. Let m be an integer coprime to n. Suppose g and h are elements of G with $g^m = h^m$. Show that $g = h$.

(b) Suppose that G is a finite simple group. Let p be the largest prime dividing $|G|$. Show that G has no proper subgroup H with $[G : H] < p$.

Problem 2. Let $T \in M_n(\mathbb{C})$ be an $n \times n$-matrix such that T^r equals the identity matrix for some integer $r \geqslant 1$.
Prove that T is conjugate to a diagonal matrix.

Problem 3. Let A be a PID and let $I_1 \supsetneq I_2 \supsetneq \dots$ be a strictly decreasing sequence of ideals. Prove that $\bigcap_n I_n = (0)$. (You should prove any non-trivial statements about PIDs that you use in this problem.)