Young Tableaux and Representation Theory

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1 Introduction

Young tableaux is a combinatorial object which has remarkable applications in representation theory and algebraic geometry. What makes it even more remarkable is that at first sight Young Tableaux does not seem as a serious mathematical object at all. In this talk I'll give a brief overview of how Young tableaux is related to representation theory of Lie algebras and quantum groups. It is a primary example of *combinatorial representation theory.*

2 Basic notions of Young Tableaux

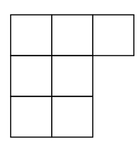
Definition 2.1 A **Young diagram** is a collection of boxes arranged in left-justified rows, with a weakly decreasing number of boxes in each row. The total number of boxes is said to be the **size** of the Young diagram.

It is clear that given a natural number n, the set of partition of n is in a one-to-one correspondence with the set of Young diagrams with size n. We use the notation $\lambda \vdash n$ or equivalently $|\lambda| = n$ when

$$\lambda = (\lambda_1, \cdots, \lambda_m), \quad \lambda_1 \geq \lambda_2 \geq \cdots \lambda_m > 0$$

is a partition of n. We write λ to denote the Young diagram whose i-th row has a length λ_i.

Example 2.2 The partition $\lambda = (3, 2, 2) \vdash 7$ corresponds to the following Young diagram.



Definition 2.3 Given a set $S = \{1, 2, \cdots, d\}$ and a Young diagram λ, a **Young tableaux** is an assignment of elements of S to each boxes of the Young diagram following these two rules:

1. assigned elements weakly increase across each row;
2. assigned elements strictly increase down each column.

We say that the Young tableaux is a tableaux **on** the diagram λ, and say that λ is the **shape** of the Young tableaux.

Example 2.4 Among the following pictures, only the leftmost one is a Young tableaux. Its shape is $\lambda = (3, 2)$.

1	2	2
2	3	

1	2	3
1	3	

2	1	3
3	3	

Definition 2.5 If a Young tableaux T has entries in the set $\{1, 2, \cdots, d\}$, its **content** is an array which specifies how many times each number is used. More precisely, it is the array $\mu(T) = (\mu_1, \cdots \mu_d)$ where μ_i is the number of i in the Young tableaux.

Definition 2.6 Given a Young diagram λ and an integer d, we can define an important polynomial called a **Schur function**, which is written as $s_\lambda(x_1, \cdots, x_d)$. Let $\mathcal{T}$ be the set of Young tableaux with a shape λ in entries in $\{1, 2, \cdots, d\}$. For each Young tableaux $T \in \mathcal{T}$ we have a monomial $x^T = \prod_{i=1}^{m} x_i^{\mu_i}$, where $\mu(T) = (\mu_1, \cdots, \mu_d)$ is the content of T. The Schur function $s_\lambda(x_1, \cdots, x_d)$ is defined to be the sum of x^T over every $T \in \mathcal{T}$. A coefficient of x^μ equals the number of Young tableaux with shape λ and content μ, which is said to be a **Kostka number** written as K_λ^μ.

Example 2.7 The following is the complete list of Young tableaux with shape $\lambda = (2, 2)$ and $d = 3$. We can read off the Schur function as $s_\lambda(x_1, x_2, x_3) = x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$.

$x_1^2x_2^2$	$x_1^2x_2x_3$	$x_1^2x_3^2$	$x_1x_2^2x_3$	$x_1x_2x_3^2$	$x_2^2x_3^2$
1 1	1 1	1 1	1 2	1 2	2 2
2 2	2 3	3 3	2 3	3 3	3 3

Remark 2.8 We can observe that a Schur function is a symmetric polynomial. It is a highly nontrivial fact. The easiest way of proving it is using *Pierri's rule*, which is a recurrence relation for Schur function in the lexicographic ordering of Young diagrams.

3 Basic notions of Lie algebra

Definition 3.1 A vector space L over a field $\mathbb{F}$, with an operation $L \times L \to L$ denoted $(x, y) \mapsto [x, y]$ and called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over $\mathbb{F}$ if the following axioms are satisfied.

1. **Bilinearity**: The bracket operation is $\mathbb{F}$-bilinear.
2. **Anti-commutativity**: $[x, y] = -[y, x]$ for every $x, y \in L$.
3. **Jacobi identity**: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for every $x, y, z \in L$.

A **Lie algebra homomorphism** $\phi : L \to L'$ between two Lie algebras L and L' is a linear map which preserves the bracket, i.e., $\phi([x, y]) = [\phi(x).\phi(y)]$ for $x, y \in L$. It is an **isomorphism** if it has an inverse.

Example 3.2 An associative $\mathbb{F}$-algebra $\mathcal{A}$ is a Lie algebra with bracket $[A, B] = AB - BA$.

Example 3.3 The vector space $\mathbb{F}^3$ with Lie bracket $[v, w] = v \times w$ is a Lie algebra.

Example 3.4 The most important examples of Lie algebras are the **general linear Lie algebra** $\mathfrak{gl}(V)$ and **classical Lie algebras** over the base field $\mathbb{C}$. Given a vector space V the set $\mathrm{End}(V)$ has a structure of an associative $\mathbb{F}$-algebra. From **Example 3.2** it has a canonical Lie algebra structure. To emphasize that we are using the Lie algebra structure, we denote it by $\mathfrak{gl}(V)$. For $n \in \mathbb{Z}_{>0}$ the $\mathfrak{gl}(n)$ refers to $\mathfrak{gl}(\mathbb{F}^n)$, and it is identified with the vector space of $\mathrm{Mat}_{n\times n}(\mathbb{F})$ with the Lie bracket $[A, B] = AB - BA$. Below is the list of classical Lie algebras. Their Lie brackets are defined as $[A, B] = AB - BA$.

1. $A_n = \mathfrak{sl}(n + 1) = \{x \in \mathfrak{gl}(n + 1) : \mathrm{tr}(x) = 0\}$
2. $B_n = \mathfrak{so}(2n + 1) = \{x \in \mathfrak{gl}(2n + 1) : x + x^T = 0\}$
3. $C_n = \mathfrak{sp}(2n) = \left\{x \in \mathfrak{gl}(2n) : J_nx + x^TJ_n = 0, J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\right\}$
4. $D_n = \mathfrak{so}(2n) = \{x \in \mathfrak{gl}(2n) : x + x^T = 0\}$

Definition 3.5 Given a Lie algebra $\mathfrak{g}$, its **Lie algebra representation** is a tuple (V, ϕ), where V is a vector space and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorhpism. We write $xv = \phi(x)(v)$ for $x \in \mathfrak{g}$ and $v \in V$ whenever the map ϕ is clear from the context. We say that the Lie algebra $\mathfrak{g}$ **acts** on the vector space V and the map $\phi(x) \in \mathfrak{gl}(V)$ is said to be an **action** of $x \in \mathfrak{g}$. If V contains a vector subspace W where each $\phi(x) : V \to V$ descends to $\phi(x) : W \to W$, then W is said to be a **subrepresentation** of V. The representation (V, ϕ) is said to be **irreducible** if V does not admit a proper nontrivial subrepresentation.

It is instructive to compare the notion of Lie algebra representation with the notion of group representation and associative $\mathbb{F}$-algebra representation. Given a group G its representation is a tuple (V, ϕ), where V is a vector space and $\phi : G \to \mathrm{GL}(V)$ is a group homomorphism. Given an associative $\mathbb{F}$-algebra A its representation is a tuple (V, ϕ), where V is a vector space and $\phi : A \to \mathrm{End}(V)$ is an $\mathbb{F}$-algebra homomorphism. Generally speaking, if we want to define a notion of representation of a certain algebraic structure, we must have a target which is a suitable subset of $\mathrm{End}(V)$ which has the same algebraic structure, and the map ϕ preserving the structure. For group representation the subset $\mathrm{GL}(V)$ is chosen because $\mathrm{End}(V)$ is not a group.

Example 3.6 Consider the Lie algebra $\mathfrak{sl}(2) = A_1$. It has a basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which satisfies relations

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

The set of finite-dimensional irreducible representations is in a one-to-one correspondence with $\mathbb{Z}_{\geq 0}$. Given $l \in \mathbb{Z}_{\geq 0}$ we can think of a vector space $V_l = \mathrm{Span}_{\mathbb{C}}\{v_0, \cdots, v_l\}$ with an action

$$Hv_i = (l - 2i)v_i, \quad Ev_i = iv_{i-1}, \quad Fv_i = (l - i)v_{i+1}.$$

We can readily confirm that this is a representation of A_1, and it is not very hard to confirm that this representation is irreducible.

Note 3.7 Given two representations $\phi_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\phi_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ we can produce a new representation by either taking a direct sum or a tensor product. For the direct sum, we can form a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2)$ by the rule $x(v_1, v_2) = (xv_1, xv_2)$. For the tensor product, we can form a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V_1 \otimes_{\mathbb{F}} V_2)$ by the rule $x(v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2$.

Remark 3.8 The choice $x(v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2$ of action for the tensor product might seem to be arbitrary at first sight. However, it can be explained in the unified framework of *Hopf algebra*. The *universal enveloping algebra* of a Lie algebra has a Hopf algebra structure.

4 Questions in the representation theory of Lie algebras

Note 4.1 From now on, we fix $\mathbb{F} = \mathbb{C}$.

Since our base field $\mathbb{C}$ is algebraically closed and is of a characteristic zero, we have the following theorem.

Theorem 4.2 (Weyl's complete reducibility theorem) Every finite-dimensional representation of finite-dimensional *semisimple Lie algebras* is a direct sum of irreducible representations.

Remark 4.3 We will not give a definition of semisimple Lie algebra here but instead list some important facts. Each finite-dimensional semisimple Lie algebra is a direct sum of *simple Lie algebras*, and we have a complete classification of simple Lie algebras. Therefore to understand the representation theory of finite-dimensional semisimple Lie algebras, it suffices to understand the representation theory of simple Lie algebras. Classical Lie algebras are simple, and there are only five exceptional simple Lie algebras other than classical Lie algebras. An interesting fact is that after we develop the theory of *root system*, the classification problem boils down into an elementary combinatorial problem which ultimately gives *Dynkin diagrams*.

Notation 4.4 For this section, let $\mathfrak{g}$ be a simple Lie algebra.

From Weyl's complete reducibility theorem, the following questions arise naturally in the representation theory of $\mathfrak{g}$. The first question is the following.

Q1: Can we completely classify the finite-dimensional irreducible representations of $\mathfrak{g}$?

We have an affirmative answer.

Notation 4.5 Let $\mathcal{C}$ be the set of finite-dimensional isomorphism classes of irreducible represenations of $\mathfrak{g}$.

The set $\mathcal{C}$ is in a one-to-one correspondence with the set of *dominant integral weights*, and we have a conceptual answer to the structure of each irreducible representation: a *maximal quotient* of a *Verma module.* However, this conceptual answer is not very helpful for explicit computations. This leads us to the following question.

Q1-1: Can we concretely understand the finite-dimensional irreducible representations of $\mathfrak{g}$?

The answer is affirmative. Surprisingly, Young tableaux enters in the description of finite-dimensional irreducible representations of simple Lie algebras. For the sake of simplicity, let us assume that $\mathfrak{g} = A_n$ for a moment. The set $\mathcal{C}$ is in a one-to-one correspondence with the set of Young diagrams with at most n rows. For a Young diagram λ the corresponding irreducible representation $V(\lambda)$ has a $\mathbb{C}$-basis $\{e_T\}_{T\in\mathcal{T}}$ where $\mathcal{T}$ is the set of Young tableaux on the Young diagram λ with entries chosen in the set $\{1, 2, \cdots, n+1\}$.

Example 4.6 Recall from **Example 3.6** that the set of finite-dimensional irreducible representations of A_1 is $\{V_l\}_{l\in\mathbb{Z}_{\geq 0}}$. The corresponding Young diagram is of a shape (l). The basis $\{v_0, \cdots, v_l\}$ of V_l is realized as the set of Young tebleaux with shape (l) and entries in $\{1, 2\}$. Specifically, the base v_i corresponds to the Young tableaux

$$\boxed{1}\boxed{1}\ldots\boxed{1}\boxed{2}\boxed{2}\ldots\boxed{2}$$

where the number of 1 is l.

A detailed account for type A Lie algebras can be found in the book [F]. For other types of simple Lie algebras, we need some generalized versions of Young tableaux. The definition of Young diagram may differ, and rules of filling out Young diagram may differ. An overview for classical Lie algebras can be found in the thesis [W], and for exceptional Lie algebras one may consult the book [C].

Let's think of another natural question arising in the study of representation theory of $\mathfrak{g}$.

Q2: Given a finite-dimensional representation V of $\mathfrak{g}$, can we find a decomposition

$$V \cong W_1^{\oplus m_1} \oplus W_2^{\oplus m_2} \oplus \cdots \oplus W_r^{\oplus m_r}$$

where $W_i \in \mathcal{C}$ and $m_i \in \mathbb{Z}_{>0}$?

Assume that we know the irreducible decomposition of two representations V_1 and V_2. Then we have an obvious answer for the irreducible decomposition of $V_1 \oplus V_2$. A more interesting problem would be to ask the irreducible decomposition of $V_1 \otimes V_2$. We can immediately reduce the problem to the situation where V_1 and V_2 are irreducible.

Q2-1: Given $V_1, V_2 \in \mathcal{C}$, can we find a decomposition

$$V_1 \otimes V_2 \cong W_1^{\oplus m_1} \oplus W_2^{\oplus m_2} \oplus \cdots \oplus W_r^{\oplus m_r}$$

where $W_i \in \mathcal{C}$ and $m_i \in \mathbb{Z}_{>0}$?

This kind of problem is called **tensor product decomposition** problem, and it is an important problem in various types of representation theory. The concept of *character* is useful in this type of problem.

Pseudo-Definition 4.7 A **character** is an invariant attached to each representation which enjoys the following properties.

1. If V_1 and V_2 are isomorphic, then $\mathrm{ch}(V_1) = \mathrm{ch}(V_2)$.
2. $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$
3. $\mathrm{ch}(V_1 \otimes V_2) = \mathrm{ch}(V_1) \times \mathrm{ch}(V_2)$
4. For distinct $V_1, \cdots, V_m \in \mathcal{C}$, the characters $\mathrm{ch}(V_1), \cdots, \mathrm{ch}(V_m)$ are linearly independent over $\mathbb{C}$.

From the properties of character, we can immediately know that the question **Q2-1** is solved if we know the characters of all irreducible representations. Again, we have a conceptual answer, namely *Weyl character formula*. However, despite its elegance the formula is not suitable for explicit computations. Moreover, if we solely rely on Weyl character formula, it is hard to give an interpretation of the components $W_1, \cdots, W_r$ and multiplicities $m_1, \cdots, m_r$. Here again Young tableaux becomes extremely handy. To simplify, let us assume $\mathfrak{g} = A_n$ for a moment. Recall that we can concretely understand irreducible representations using Young tableaux. We can reformulate the question as the following.

> **Q2-2**: Can we read off the character of $V(\lambda) \in \mathcal{C}$ from the combinatorial model given by Young tableaux? Given $V(\lambda_1), V(\lambda_2) \in \mathcal{C}$ can we find a combinatorial interpretation to the decomposition $\mathrm{ch}(V(\lambda_1)) \times \mathrm{ch}(V(\lambda_2)) = \sum_\nu m_\nu \mathrm{ch}(V(\nu))$?

Again we have an affirmative answer. Recall that for an irreducible representation $V(\lambda)$ we have a basis $\{e_T\}_{T\in\mathcal{T}}$ parametrized by Young tableaux on the shape λ. The contribution of e_T on $\mathrm{ch}(V(\lambda))$ is solely determined by its content, so it turns out that the Schur function s_λ with appropriate insertion of variables equals the character of $V(\lambda)$. Therefore for the tensor decomposition problem we only have to understand the decomposition $s_{\lambda_1}s_{\lambda_2} = \sum_\lambda c^\nu_{\lambda_1\lambda_2} s_\nu$. From the combinatorics of Young tableaux, it turns out that the coefficient $c^\nu_{\lambda_1\lambda_2}$ is a *Littlewood-Richardson number*. It is the number of ways of expressing a given Young tableaux T on the shape ν by the *product* $T_1 * T_2$ of Young tableaux T_1, T_2 on the shapes λ_1, λ_2, respectively. It does not depend on the choice of T. There has been extensive research on effective algorithms of computing Scuhur functions (or equivalently Kostka numbers) and Littlewood-Richardson numbers.

Representation Theory	**Combinatorics**
labelling of irreducible representations	Young diagram
basis of irreducible representation	Young tableaux
weight of a base	content of a Young tableaux
dimension of weight space	Kostka number
character of irreducible representation	Schur function
multiplicity in tensor product decomposition problem	Littlewood-Richardson number

5 A Brief Remark on Quantum Groups

One area that combinatorics of Young tableaux manifests itself is the representation theory of *quantum groups*. Quantum groups and their representations are quite complicated objects, and you will probably find yourself struggling even when you try to do some basic computations. However, the theory of *crystal bases* developed by M. Kashiwara and G. Lusztig in early 1990's greatly helps us analyze those objects. To define what a quantum group is, we have to start from the notion of *universal enveloping Lie algebra*.

Definition 5.1 Let $\mathfrak{g}$ be a Lie algebra. The **universal enveloping algebra** $U\mathfrak{g}$ is the unital associative $\mathbb{F}-$algebra with a Lie algebra homomorphism $\iota : \mathfrak{g} \to U\mathfrak{g}$ which satisfies the following universal property:

> for every Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ to a unital associative algebra A there exists a unique algebra homomorphism $\widetilde{\phi} : U\mathfrak{g} \to A$ such that $\widetilde{\phi} \circ \iota = \phi$.

The definition above of universal enveloping algebra might seem too abstract, but there is a conceptual way of understand it. Lie algebra is not an algebra we are familiar with, in the sense that it does not satisfy associativity. A universal enveloping algebra is a unital associative algebra which can substitute Lie algebra without losing any information. More precisely, the representation theory of $U\mathfrak{g}$ as an associative algebra is the same as the representation theory of $\mathfrak{g}$ as a Lie algebra. Moreover, we have a nice basis of the universal enveloping algebra, namely *PBW basis*.

Pseudo-Definition 5.2 A **quantum group** $U_q\mathfrak{g}$ is a q-analogue (or q-deformation) of the universal enveloping algebra $U\mathfrak{g}$.

Remark 5.3 In fact quantum group is not defined for arbitrary Lie algebras. Instead, it is defined for *symmetrizable Kac-Moody algebras*. We won't explain the definition of symmetrizable Kac-Moody algebras, but it covers almost every Lie algebras we would be interested in, including finite-dimensional semisimple Lie algebras and affine Lie algebras. Note that affine Lie algebras are of infinite-dimension.

Notation 5.4 For this section let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra.

There is a category $\mathcal{O}^q_{\text{int}}$ of *integrable representations* of a quantum group $U_q\mathfrak{g}$. It is a full subcategory of the category of representations of $U_q\mathfrak{g}$ which behaves nicely: it is closed under taking direct sum, tensor product, and subrepresentation. A highly nontrivial fact is that this category enjoys complete reducibility.

Note 5.5 Throughout this section, a representation is an object of $\mathcal{O}^q_{\text{int}}$.

Theorem 5.6 (G. Lusztig) Every representation can be written as a direct sum of irreducible representations.

Therefore it is natural for us to ask the same questions as before. For **Q1-1** we have both a good news and bad news. We have a *global bases* of both representations and the *negative part* $U_q^-\mathfrak{g}$ of a quantum group. The bad news is that we cannot easily write down the action of quantum groups using the global bases, even though it substantially simplifies the action. There are two good news. First one is that this basis is defined for arbitrary symmetrizable Kac-Moody algebra. Recall that in the previous section we only had models for finite-dimensional semisimple Lie algebras. The second one is that we now have richer structures which are very suitable for combinatorial arguments.

Pseudo-Definition 5.7 For a representation $V \in \mathcal{O}^q_{\text{int}}$ or a negative part $U_q^-\mathfrak{g}$ of a quantum group we can attach an invariant **crystal graph** to each of them. It is a colored directed graph whose vertices are given by 'freezing' global bases to the limit $q \to 0$ and arrows are given by freezing the action of generators of the quantum group.

Remark 5.8 We use the expression 'freezing' for the $q \to 0$ limit because in the physics context of quantum groups the parameter q of deformation represents the temperature.

Crystal graphs enjoy the following remarkable properties.

1. Two isomorphic representations have isomorphic crystal graphs.
2. They are stable under taking direct sum and tensor product of representations.
3. They stable under the canonical projection $U_q^-\mathfrak{g} \to V$ where V is an irreducible representation.
4. The set of connected components of a crystal graph is in a one-to-one correspondence with irreducible components of the given representation.

Given crystal graphs G_V and G_W of two representations V and W respectively, the crystal graph of $V \oplus W$ is the disjoint union of G_V and G_W. Moreover, we can easily obtain the crystal graph of $V \otimes W$ by applying the *tensor product rule* to the graphs G_V and G_W. These combined with *Kashiwara embedding* gives us a good framework for making a combinatorial description of crystal graphs. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ one obtains a nice Young tableaux description of crystal graphs for irreducible representations. From the last property of crystal graphs the tensor product decomposition problem **Q2-2** boils down to a combinatorics of Young tableaux.

Example 5.9 **Figure 1** and **Figure 2** are examples of Young tableaux description of crystal graphs. **Figure 1** is a crystal graph for the irreducible representation of $\mathfrak{g} = \mathfrak{sl}_3$ with the highest weight $2\omega_2$, and **Figure 2** is the crystal graph for the negative part $U_q^- \mathfrak{g}$ of the quantum group. Note that the crystal graph of $B(\infty)$ contains the crystal graph of $B(\lambda)$ as a subgraph. (see the colored vertices in **Figure 2** and compare with **Figure 1**) This follows from the property of crystal graph that they are stable under the canonical projection $U_q^- \mathfrak{g} \to V$.

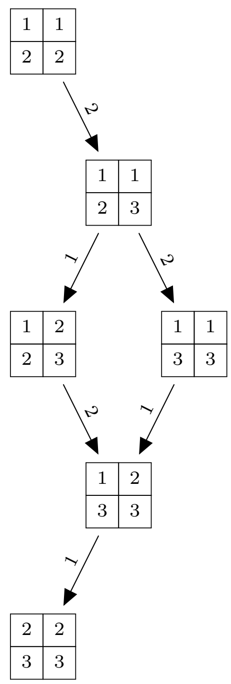


Figure 1: The crystal graph $B(\lambda)$ for $\lambda = 2\omega_2$, $\mathfrak{g} = \mathfrak{sl}_3$

References

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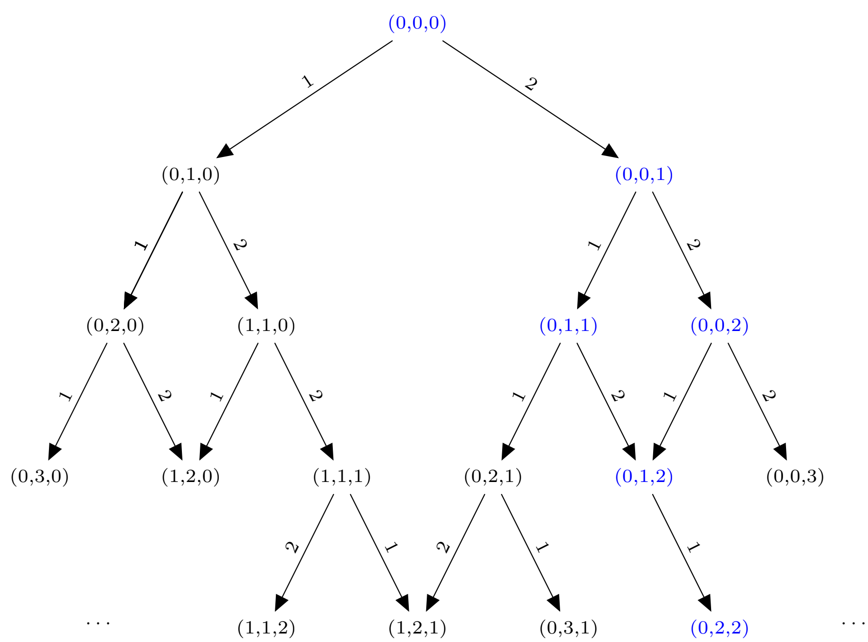

Figure 2: The crystal graph $B(\infty)$ for $\mathfrak{g} = \mathfrak{sl}_3$