Henry Segerman — *Research Statement*

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My research is in the areas of three-dimensional topology and hyperbolic geometry, particularly in tackling concrete problems, often from a combinatorial point of view.

Given a three-manifold M with torus boundary components, in general many different geometric structures can be put onto it. One way to describe a geometric structure is to cut M into ideal[1] tetrahedra (i.e. choose a tetrahedralisation $\mathfrak{T}$ of M), and then specify the shape of each tetrahedron in hyperbolic space ($\mathbb{H}^3$). It turns out that each shape can be described by a single complex number[2] assigned to a dihedral angle of the tetrahedron, and consistency requirements on the shapes for the tetrahedra to fit together nicely can be expressed as polynomial equations in these 'complex angles'. The resulting variety is called the deformation variety and was described by Thurston [11] in 1980.

My work has centred on furthering our understanding of the deformation variety, which is a more concrete and visual object than other spaces we have to organise geometric structures[3], but has some problematic features. In particular, the deformation variety depends crucially on the tetrahedralisation $\mathfrak{T}$: there may be entire components of the deformation variety with one tetrahedralisation for which no corresponding component exists in the deformation variety with a second tetrahedralisation. The problem arises if the geometry of the manifold forces some tetrahedron to have two vertices at the same point on the boundary of $\mathbb{H}^3$, so that the corresponding complex angle (cross-ratio) is infinite. This kind of situation cannot be represented by a solution to the polynomial equations defining the deformation variety.

In my most recent paper, *A generalisation of the deformation variety* [9], I introduced a generalisation of the deformation variety, which again consists of assignments of complex variables to certain dihedral angles subject to polynomial equations, but together with some extra combinatorial data concerning degenerate tetrahedra. This "extended deformation variety" deals with many situations that the deformation variety cannot. To a large extent it solves the problem of dependence on the tetrahedralisation, and should be very useful in making the deformation variety more like the representation variety, while preserving the visual and combinatorial aspects of working with tetrahedralisations. The main result is:

Theorem. *Let M be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori and suppose that M admits an ideal tetrahedralisation. Then there exists an ideal tetrahedralisation of M such that every irreducible representation whose image is not a generalised dihedral group can be recovered from the associated extended deformation variety.*

Tillmann [12] notes that if M admits a complete hyperbolic structure of finite volume and a (standard) deformation variety for M with some tetrahedralisation is non-empty, then it has a component which corresponds to the component of the representation variety containing the complete structure. However, individual representations may be missing from this correspondence, and it says nothing about other components. The extended deformation variety fills these gaps, acting in some ways like a compactification.

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[1] An ideal tetrahedron is a tetrahedron without its vertices. We think of the vertices as being infinitely far away, just as the 'endpoints' of $\mathbb{R}$ are infinitely far away.

[2] The complex number is a cross-ratio of the four ideal vertices on the boundary of $\mathbb{H}^3$, viewed as $\mathbb{C} \cup \{\infty\}$.

[3] Namely, the representation and character varieties. The representation variety is the set of representations ρ from the fundamental group of M to the set of isometries of $\mathbb{H}^3$, and is a more intrinsic but less concrete object than the deformation variety. The character variety is, roughly, the set of traces of elements of the representation variety.

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Degenerating geometric structures

As we deform the geometric structure on a manifold (by following a path in one of these varieties) it can happen that the length of some path γ through the manifold diverges to infinity. There is a well-known correspondence between points of the variety at which this behaviour occurs (ideal points of the variety) and incompressible surfaces[4] within the manifold: The manifold splits apart along surfaces through which γ passes as the distances between regions of the manifold to each side of the surface diverge. This correspondence was introduced by Culler and Shalen in a seminal series of papers, giving a method to construct incompressible surfaces given ideal points[5]. The question naturally arises as to the extent that all incompressible surfaces come from ideal points.

My thesis work, and my paper *Detection of incompressible surfaces in hyperbolic punctured torus bundles* [8] answered this question in the context of punctured torus bundles[6], and along the way gave a concrete geometric understanding of the splittings of these manifolds. The main result is:

Theorem. *All incompressible surfaces in hyperbolic punctured torus bundles over the circle can be constructed from ideal points of the deformation variety.*

Not much is known about this question in general, although it is believed that "most" incompressible surfaces in "large" manifolds should not be detectable (i.e. constructable from ideal points). Results on this are often stated in terms of slopes on the boundary torus of the 3-manifold being **strongly detected**, meaning that it is the boundary slope of an incompressible surface that comes from an ideal point such that no closed surface comes from the same ideal point. In particular, Ohtsuki [6] shows that all boundary slopes of incompressible surfaces in 2-bridge knots are strongly detected, but that not every incompressible surface can be obtained from the construction. Given the similarities between punctured torus bundles and 2-bridge knots, it seemed likely that something could be said about this issue for torus bundles, although his techniques are very different from mine.

In 1982, Floyd and Hatcher [3], and Culler, Jaco and Rubinstein [4] classified the incompressible surfaces in punctured torus bundles, so all that is needed is to show that each of these is detected. I used an alternate version of the Culler-Shalen construction [2] due to Yoshida [14], who works with the deformation variety and uses combinatorial results of Neumann and Zagier [5]. Starting from an ideal point of the deformation variety, Yoshida constructs a surface from twisted squares (see figure 1) placed in tetrahedra that degenerate as we approach the ideal point. The main idea of my paper is to run this construction in reverse: we isotope an incompressible surface to get it into the form that Yoshida's construction outputs (which involves some tricky combinatorial arguments), then we run Yoshida's construction backwards to get the relative rates of degeneration of the different tetrahedra we are supposed to see as we approach our (hopefully) ideal point. Showing that these rates correspond to an actual ideal point of the deformation variety involves a number of ideas, including some algebraic geometry.

[4]Incompressible surfaces are intrinsic to the topological structure of the 3-manifold, and are important to the study of 3-manifolds in general.

[5]Strictly, ideal points of the character variety rather than ideal points of the deformation variety. The character variety is closely related to the representation variety, and removes some "slack" in the representation variety due to conjugate representations.

[6]Punctured torus bundles are a well studied class of three-manifolds with (in particular) well understood tetrahedralisations.

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We would also like to be able to say that the ideal points of the deformation variety that we find correspond to ideal points of the representation variety, and that the twisted squares surfaces we generate using Yoshida's construction can also be generated from the Culler-Shalen construction. Tillmann [12] proves this in general for spun-normal surfaces generated from ideal points of the deformation variety.

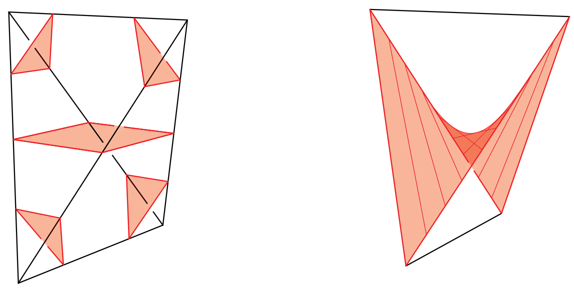

Figure 1: Spun-normal and twisted squares surfaces are two kinds of combinatorially defined surfaces embedded in a manifold with tetrahedralisation. The former are made up of quadrilateral and triangle parts (on the left), the latter of twisted squares (on the right). The relation between the two types of surface is quite complicated.

In my paper *On spun-normal and twisted squares surfaces* [10] I investigated the relationship between twisted squares surfaces and spun-normal surfaces, and using Tillmann's analogous result for spun-normal surfaces, showed the following:

Theorem. *Let M be an oriented 3-manifold with ∂M a union of tori with ideal triangulation $\mathcal{T}$ and T a two-sided twisted squares surface obtained via Yoshida's construction from an ideal point of the deformation variety $\mathfrak{D}(M, \mathcal{T})$ which corresponds to an ideal point of the character variety. Then any incompressible surface obtained from T by compressions is detected by the character variety.*

This paper then extends the result of [8] from ideal points of the deformation variety and Yoshida's construction to ideal points of the character variety and the Culler-Shalen construction.

Other Interests

Yet another way of organising geometric structures in the case of a 3-manifold M with a single cusp is the Dehn surgery space of M. In the case of a manifold that has a complete hyperbolic structure, the Dehn surgery space can be seen as a subset of the Reimann sphere, with points corresponding to geometric structures with particular behaviours on the boundary torus. The point at ∞ corresponds to the complete structure, and the geometry corresponding to an integral point (x, y) is such that moving x times around the meridian and y times around the longitude of the boundary torus is trivial. There is a way to extend to non-integral points. By a result of Thurston [11], Dehn Surgery space contains a neighbourhood of ∞.

The four vertices in the figure at $(\pm 4, \pm 1)$ correspond to Dehn fillings of the manifold such that the boundary curves of incompressible surfaces in the manifold are filled in. In some sense, the Dehn surgery space is a projection of the deformation (or representation) variety, and so we see ideal points showing up

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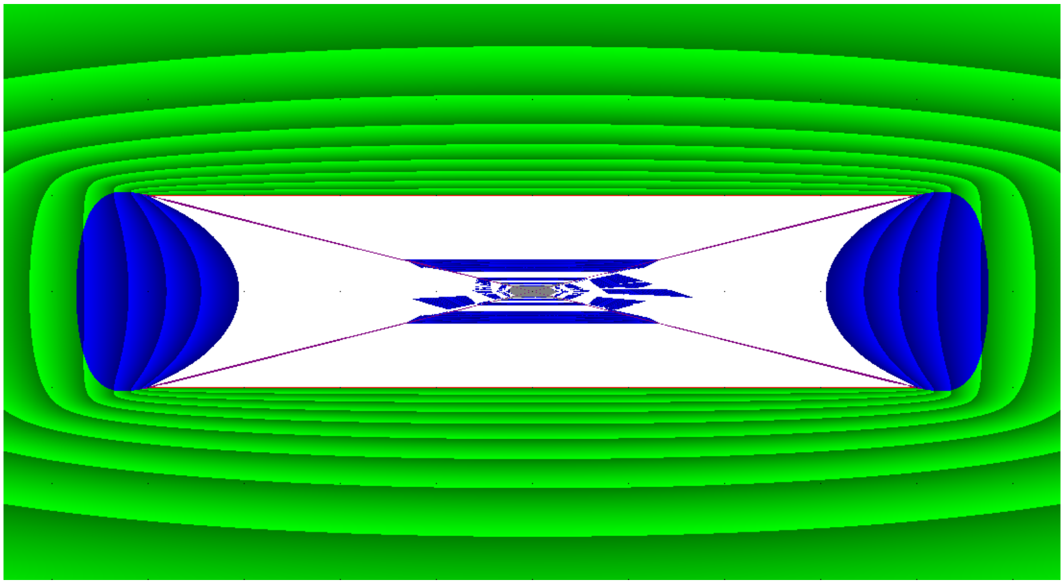

Figure 2: Conjectural boundary of Dehn surgery space for the figure 8 knot complement (which can also be seen as a punctured torus bundle). Tracy Hall, Saul Schleimer and I generated this image using SnapPeaPython [1] by Marc Culler and Nathan Dunfield, who built on SnapPea [13] by Jeff Weeks. The green and blue areas are hyperbolic structures with the shading corresponding to volume, blue with some negatively oriented tetrahedra and green with all positively oriented tetrahedra. Red corresponds to flat structures and other colours are outside of Dehn surgery space. Black points mark the integer coordinates.

on the boundary of Dehn surgery space. The (red) straight line at the top of the Dehn surgery boundary picture consists of $\widetilde{SL_2\mathbb{R}}$ structures (and a Sol structure at $(0, 1)$). One of my interests has been in trying to understand the collapse of (incomplete) hyperbolic structures corresponding to Dehn surgery coordinates just above this line as we move to the line, using the deformation variety. See the left diagram of figure 3.

I am currently working on a project with João Miguel Nogueira, studying incompressible (but not boundary incompressible) surfaces in the genus two handlebody. We show that for every compact surface with boundary, orientable or not, there is an incompressible embedding of the surface into the genus two handlebody. In the orientable case the embedding can be either separating or non-separating. We extend a result of Qiu [7], who shows the existence of orientable incompressible surfaces with any genus but only two boundary components.

I have a continuing interest in the use of modern computer graphics software for effective communication and visualisation in mathematics, and topology in particular. Ken Baker, Loretta Bartolini, Jesse Johnson and I are currently in the process of organising a workshop on the use of graphics in low dimensional topology, to be paired with a topology research conference. The workshop will introduce topologists to current software and encourage exchange of techniques and methods in 2D, 3D, animation and other areas of computer graphics.

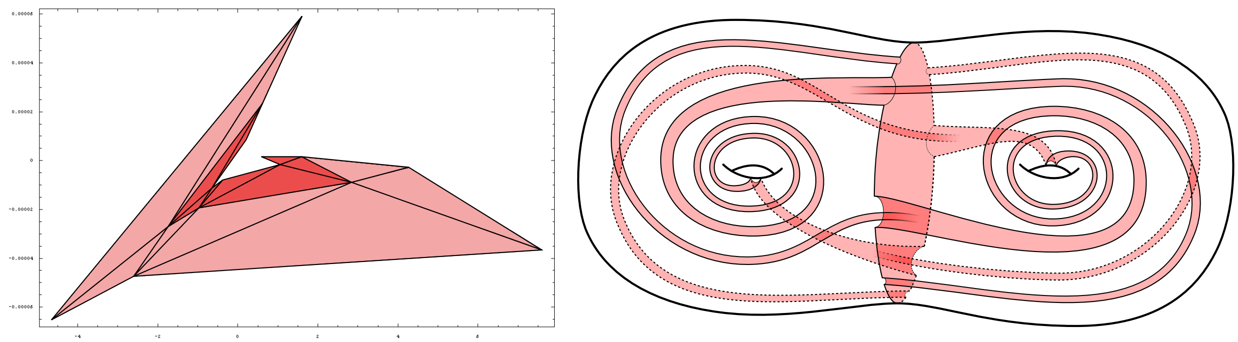

Figure 3: On the left: tetrahedra flatten out as we approach the boundary of Dehn Surgery space and the hyperbolic volume approaches zero. The tetrahedra are viewed from infinity in the upper half space model of $\mathbb{H}^3$ so appear as triangles. The vertical axis is at a much larger scale than the horizontal and so the tetrahedra are very nearly flat. On the right: an incompressible surface in a handlebody of genus two.

Future Directions

The proof of [8] goes through by demonstrating that each incompressible surface has an ideal point of the deformation variety corresponding to it. The deformation variety depends on the tetrahedralisation, and so it was a stroke of luck that the tetrahedralisation I used for each punctured torus bundle was such that the deformation variety with respect to that tetrahedralisation had all of the required ideal points. If I had been unable to find corresponding ideal points we would not know if there were no such ideal point to be found (as in the case for 2-bridge knots, [6]), or if we just had a bad choice of tetrahedralisation.

The extended deformation variety solves the problem of not knowing if the tetrahedralisation is bad or not, and so an obvious next step to take is to figure out how to compactify the extended deformation variety, in a way similar to how Yoshida [14] and Tillmann [12] compactify the (standard) deformation variety. This should again lead to a construction of incompressible surfaces, and in particular, an algorithmic way to detect them. Closed incompressible surfaces are particularly difficult to identify using existing techniques (neither Tillmann nor Yoshida tackles them), but this work should provide a different approach which does not suffer from these issues.

Tropical geometry may be the correct framework in which to describe this compactification, and I am working with Eric Katz on understanding this. We also plan to develop another generalisation of the deformation variety, which we call the "universal deformation variety". This object avoids the dependence on a particular tetrahedralisation by recording every possible cross ratio between any four cusps in the universal cover of the manifold, rather than just those corresponding to tetrahedra of a given tetrahedralisation. The extended deformation variety for such a tetrahedralisation would then be a finite dimensional projection of the universal deformation variety.

Aside from the compactification issue, the very concrete picture of the extended deformation variety should be a useful tool in investigating components of the representation variety other than the component containing the complete structure. Very little is known about these other components. In some examples one can see other components as a result of the fundamental group of the manifold factoring through the fundamental group of some other manifold, and it may be that more generally, some sort of topologically meaningful interpretation could be made of the geometric structures on every component.

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