Homework 11

Due Friday May 1

1. (Ch 6, Question 5.) Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$ on $[a, b]$? Does the answer change if we assume that $f^3 \in \mathcal{R}$ on $[a, b]$? *Hint:* Consider theorem 6.11.

2. (Ch 6, Question 6.) Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on [0,1] which is continuous at every point outside of P. Prove that $f \in \mathcal{R}$ on $[0, 1]$. *Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in theorem 6.10.

3. (Ch 6, Question 8.) Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

   if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

   Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

   converges if and only if

$$\sum_{n=1}^\infty f(n)$$

   converges. (This is the so-called "integral test" for convergence of series.)

4. (Ch 7, Question 1.) Prove that every uniformly convergent sequence of bounded functions $\{f_n : E \to \mathbb{R}\}$ is uniformly bounded. In other words, given that there exist M_n so that $|f_n(x)| < M_n$ for all x, and that $\{f_n\}$ converges uniformly on E to a function $f : E \to \mathbb{R}$, prove that there exists a number M such that $|f_n(x)| < M$ for all $x \in E, n \in \mathbb{N}$.