Homework 12

Due Friday May 8

1. (Ch 7, Question 2.) If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

2. Construct a convergent (*not* uniformly convergent) sequence of Riemann integrable functions $f_n : \mathbb{R} \to \mathbb{R}$ such that the limit function is not Riemann integrable. *Hint:* what functions do we know that are not Riemann integrable? How could you "build" it from integrable functions?

3. (Ch 7, Question 14.) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for all t and $f(t) = 0$ for $t \in [0, \frac{1}{3}]$, $f(t) = 1$ for $t \in [\frac{2}{3}, 1]$.

   Put $\Phi(t) = \big(x(t), y(t)\big)$, where

   $$x(t) = \sum_{n=1}^{\infty} \frac{f(3^{2n-1}t)}{2^n}, \qquad y(t) = \sum_{n=1}^{\infty} \frac{f(3^{2n}t)}{2^n}$$

   Prove that $\Phi : [0,1] \to [0,1] \times [0,1]$ is *continuous* and *onto*.

   Hint: Each $(x_0, y_0) \in [0,1] \times [0,1]$ has the form

   $$x_0 = \sum_{n=1}^{\infty} \frac{a_{2n-1}}{2^n}, \qquad y_0 = \sum_{n=1}^{\infty} \frac{a_{2n}}{2^n}$$

   where each a_i is 0 or 1 (i.e. in infinite binary expansion). If

   $$t_0 = \sum_{n=1}^{\infty} \frac{2a_i}{3^{i+1}}$$

   show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0, y(t_0) = y_0$.