# Homework 4

Due Friday Feb 27

1. (Ch 2, Question 10.) Let $X$ be an infinite set. For $p, q \in X$ define:

$$d(p,q) = \begin{cases} 1 \text{ (if } p \neq q) \\ 0 \text{ (if } p = q) \end{cases}$$

   Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact? Prove your assertions.

2. Let $E \subset \mathbb{R}$ be given by $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Show that $E$ is not compact.

3. Show that the union of a finite number of compact subsets of a metric space is compact.

4. (Ch 2, Question 19.)

   (a) If $A$ and $B$ are disjoint closed sets in some metric space $X$ prove that they are separated.

   (b) Prove the same for disjoint open sets.

   (c) Fix $p \in X, \delta > 0$, define $A = \{q \in X \mid d(p,q) < \delta\}$, $B = \{q \in X \mid d(p,q) > \delta\}$. Prove that $A$ and $B$ are separated.

   (d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c), and the fact that intervals of $\mathbb{R}$ are uncountable.

5. (Ch 3, Question 1.) Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?