Homework 5

Due Friday Mar 6

1. Prove that for a sequence $\{p_n\}$ in a metric space X, $\lim_{n\to\infty} p_n = p$ if and only if the sequence $p_1, p, p_2, p, p_3, p, \ldots$ is convergent.

2. (Ch 3, Question 20.) Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

3. (Ch 3, Question 21.) Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$ and if
$$\lim_{n\to\infty} \text{diam } E_n = 0$$
then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

4. (Ch 3, Question 23.) Supose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n,
$$d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$
It follows that
$$|d(p_n, q_n) - d(p_m, q_m)|$$
is small if m and n are large.

5. Prove that any sequence in $\mathbb{R}$ has a monotonic subsequence. *Hint:* This is easy if there exists a subsequence with no least term, hence we may suppose that each subsequence has a least term.