

Homework 6

Due Friday Mar 13

1. (Ch 3, Question 4.) Find the upper and lower limits of the sequence $\{s_n\}$ defined by:

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}$$

2. Suppose that $\{a_n\}$ is a bounded sequence of real numbers. Show that:

$$\limsup_{n \rightarrow \infty} a_n = \sup\{x \in \mathbb{R} \mid a_n > x \text{ for an infinite number of integers } n\}$$

3. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers. Show that:

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

4. (Ch 3, Question 8.) If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges. *Hint:* there is some N_0 such that for all $n > N_0$ the sign of b_n is the same (why?).

5. (Ch 3, Question 14.) If $\{s_n\}$ is a complex sequence, define its *arithmetic means*

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n + 1}$$

- (a) If $\lim_{n \rightarrow \infty} s_n = s$, prove that $\lim_{n \rightarrow \infty} \sigma_n = s$.
- (b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim_{n \rightarrow \infty} \sigma_n = 0$.
- (c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim_{n \rightarrow \infty} \sigma_n = 0$?