Homework 7

Due Friday Mar 27

1. (Ch 3, Question 9.) Find the radius of convergence of each of the following power series:

   (a) $\sum n^3 z^n$

   (b) $\sum \frac{2^n}{n!} z^n$

   (c) $\sum \frac{2^n}{n^2} z^n$

   (d) $\sum \frac{n^3}{3^n} z^n$

   Hint: sometimes it may be easier to use the ratio test rather than the root test.

2. Show that a power series $\sum_{n=0}^{\infty} c_n x^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} c_{n+m} x^n$ for any fixed positive integer m.

3. (a) Show that

   $$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   has infinite radius of convergence.

   (b) Use binomial expansion to show that

   $$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

   Hint: you can show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely and use theorem 3.50.

   (c) Show that for $m \in \mathbb{N}$, $e^m = \sum_{n=0}^{\infty} \frac{m^n}{n!}$ (note that we set $0^0 = 1$ here, and take 3.30 for your definition of e).

4. For $z \in \mathbb{C}$ we define

   $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
   $$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$
   $$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

   (a) Show that $\cos z$ and $\sin z$ have infinite radius of convergence.

   (b) Show that $e^{iz} = \cos z + i \sin z$

   (c) Show that:

      i. $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

      ii. $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

      iii. $\cos^2 z + \sin^2 z = 1$