Homework 8

Due Friday Apr 3

1. (Ch 4, Question 7.) If $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function h whose domain is E and $h(p) = f(p)$ for $p \in E$. Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy^2}{(x^2+y^4)} & \text{if } (x,y) \neq (0,0) \end{cases}$$

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy^2}{(x^2+y^6)} & \text{if } (x,y) \neq (0,0) \end{cases}$$

   Prove that f is bounded on $\mathbb{R}^2$ (*Hint:* consider $(x - y^2)^2$), g is unbounded in every neighbourhood of $(0,0)$ and that f is not continuous at $(0,0)$. Prove however that the restrictions of both f and g to any straight line in $\mathbb{R}^2$ are continuous.

2. (Ch 4, Question 1.) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

   for each $x \in \mathbb{R}$. Does this imply that f is continuous?

3. (Ch 4, Question 2.) If $f : X \to Y$ is continuous prove that $f(\overline{E}) \subset \overline{f(E)}$ for every $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

4. (Ch 4, Question 3.) Let $f : X \to \mathbb{R}$ where X is a metric space and f is continuous. Let $Z(f)$ (the **zero set of** f) be $\{x \in X \mid f(x) = 0\}$. Prove that $Z(f)$ is closed.

5. (Ch 4, Question 8.) Let E be a bounded subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ be uniformly continuous on E. Show that $f(E)$ is bounded.