Homework 9

Due Friday Apr 10

1. (Ch 4, Question 14.) Let $I = [0, 1]$ be the closed unit interval. Suppose that $f : I \to I$ is continuous. Prove that $f(x) = x$ for at least one $x \in I$. *Hint:* Consider the function $g(x) = f(x) - x$.

2. (Ch 4, Question 18.) Every rational number x can be written in the form $x = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and have no common divisors. When $x = 0$, we take $n = 1$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

   Show that f is continuous at every irrational point and that f is discontinous at every rational point.

3. (Ch 5, Question 1.) Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

   for all $x, y \in \mathbb{R}$. Prove that f is constant.

4. (Ch 5, Question 13.) Suppose $a, c \in \mathbb{R}$, $c > 0$ and $f : [0, 1] \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} |x|^a \sin(|x|^{-c}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Prove the following:

   (a) f is continuous if and only if $a > 0$.

   (b) $f'(0)$ exists if and only if $a > 1$.

   (c) f' is bounded if and only if $a \geq 1 + c$.

   You may assume that the derivative of $\sin(x)$ is $\cos(x)$ and other general knowledge about sin and cos.