

SUBGROUP SEPARABILITY AND HOMOLOGY COVERS OF SURFACES

ERIC KATERMAN

ABSTRACT. We define a strong form of subgroup separability, which we call RS separability, and we use this to combine LERF and Agol’s RFRS condition on groups into a property called LVRSS. We show that some infinite classes of groups that are known to be both subgroup separable and virtually RFRS are also LVRSS. We also provide evidence for the naturalness of RS separability and LVRSS by showing that they are preserved under various operations on groups.

1. INTRODUCTION

Let G be a group. A subgroup $H \leq G$ is **separable in G** if H is equal to the intersection of all subgroups of finite-index in G containing H . A group G is **residually finite (RF)** if the trivial subgroup is separable in G . G is **locally extended residually finite (LERF)** if all finitely generated subgroups are separable in G . Subgroups of RF and LERF groups are RF and LERF, respectively. Also, both RF and LERF are virtual properties: if $H \leq_f G$ is a subgroup of finite index, and K is separable in H , then K is separable in G .

Many groups are known to be RF: for example, Mal’cev’s Theorem (sometimes called Selberg’s Lemma) asserts that all finitely generated linear groups are RF. LERF is a much stronger property—for example, the linear group $SL_n\mathbb{Z}$ is not LERF for any $n \geq 3$ [12]. However, several important classes of infinite groups are LERF: a celebrated theorem of Marshall Hall Jr. is that free groups are LERF [8]; a well-known theorem of Peter Scott asserts that the fundamental group of a closed surface is LERF [14]; and Henry Wilton has recently shown that fully residually free groups (also known as limit groups; see Section 2.1) are LERF as well [16]. Compact Seifert fiber spaces are LERF ([14] and [15]). An important class of hyperbolic 3-manifold groups that are LERF is the class of Bianchi groups, $PSL(2, \mathcal{O}_d)$; see [12].

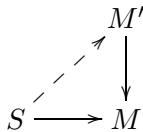
The main connection between subgroup separability and topology is Scott’s criterion (see [14], and subsections 4.2 and 5.2 below), which allows one to promote π_1 -injective immersions to embeddings in finite-sheeted covers. Indeed, some of our motivation for studying these topics comes from hyperbolic 3-manifolds: suppose M is a closed, hyperbolic 3-manifold, so

$$M = \mathbb{H}^3/\Gamma$$

where Γ is a discrete, torsion-free subgroup of $\text{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. The Virtual Haken Conjecture states that there is a finite-sheeted cover $M' \rightarrow M$ and a closed, homeomorphically embedded, π_1 -injective surface $S \hookrightarrow M'$ with $\chi(S) < 0$. There is a famous two-step program to approach this conjecture:

- (1) **The Surface Subgroup Conjecture:** There exists a closed hyperbolic surface group $\pi_1 S \leq \pi_1 M$. This yields an immersion $S \looparrowright M$.
- (2) **The LERF conjecture:** $\pi_1 M$ is LERF.

Scott’s criterion yields an embedding of S in a finite-sheeted cover $M' \rightarrow M$:



The surface subgroup conjecture was proven in 2009 by Kahn-Markovich [10], but as of the writing of this paper, the LERF conjecture remains open.

As mentioned above, RF and LERF are preserved by subgroups and supergroups of finite index. These properties are also preserved under free product. An important distinction is that RF is preserved under direct product, but LERF is not; for example, F_2 , the nonabelian free group on two generators is LERF, but $F_2 \times F_2$ has unsolvable generalized word problem, and so is not LERF.

1.1. Main results. In [2], Agol defines a property of groups called RFRS, and he proves that if the fundamental group of an irreducible 3-manifold M has this property, then M is virtually fibered, that is, M has a finite-sheeted cover $M' \rightarrow M$ such that M' is a fiber bundle over S^1 . He shows that finitely generated right-angled Coxeter groups are virtually RFRS, and it follows that closed surface groups are virtually RFRS as well. Motivated by the theory of subgroup separability and Agol's RFRS condition on groups, we make the following definition in Section 2:

Definition 1.1. A subgroup $H \leq G$ is **RS-separable** (denoted $H \leq_{RS} G$) if there is a descending tower of subgroups $\cdots \triangleleft_f G_2 \triangleleft_f G_1 \triangleleft_f G_0 := G$ whose intersection is H and

$$G_{i+1} \geq (G_i)_r^{(1)} = \{x \in G_i \mid x^d \in G_i^{(1)}, \text{ some } 0 \neq d \in \mathbb{Z}\}$$

where $G_i^{(1)}$ denotes the commutator subgroup of G_i . The collection $\{G_i\}$ is called an **RS tower over H** .

A group G is **residually finite and rationally solvable (RFRS)** if $1 \leq_{RS} G$; see [2]. Agol's primary interest is in groups that are **virtually RFRS**, that is, those groups G that contain a subgroup of finite index $K \leq_f G$ such that $1 \leq_{RS} K$. Thus, if $H \leq G$, we will say that H is **virtually RS-separable** in G (denoted $H \leq_{VRS} G$) if there is a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} K$. We will call a group G **locally virtually RS-separable (LVRSS)** if every finitely generated subgroup $H \leq G$ satisfies $H \leq_{VRS} G$. This is a very strong property; for example, LVRSS implies both virtually RFRS and LERF (see Proposition 2.2), so in particular, if a hyperbolic 3-manifold M has LVRSS fundamental group, then, by recent work of Agol [1] using work of Haglund and Wise [7], M is virtually fibered, has large fundamental group, and is cubulated.

As an aside, we mention that the subgroup $G_r^{(1)} \leq G$ appears elsewhere in the literature as the first term in the rational derived series of G ; see, for example, [4], [5], and [9].

Agol proves that the trivial subgroup of a surface group is virtually RS-separable [2]. One of the main results of this paper is that, in fact, all finitely generated subgroups of surface groups are virtually RS-separable:

Theorem A. *Let $\Sigma = \Sigma_{g,n}$ be a compact, orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. Then $\pi_1(\Sigma)$ is LVRSS.*

The proof of Theorem A in the case that $n \geq 1$ appears in Section 4 and uses a theorem on bases of subgroups of free groups due to Federer and Jónsson [6]. The main technical tool that we use to prove the $n = 0$ case of Theorem A in Section 5 is the following geometric result, which is reminiscent of the Federer and Jónsson theorem:

Theorem B. *Let (Σ, x) be an oriented hyperbolic surface with basepoint x . Let A_Σ denote the set of geodesic loops based at x that are not homotopic to a product of shorter loops based at x . Then:*

- (1) *For every $\alpha \in A_\Sigma$, α is simple.*
- (2) *For every $\alpha, \beta \in A_\Sigma$, $\alpha \cap \beta = \{x\}$.*
- (3) *$\langle [A_\Sigma] \rangle = \pi_1(\Sigma, x)$.*

In particular, if $\pi_1(\Sigma, x)$ is finitely generated, then A_Σ is finite.

This collection of simple, minimally intersecting loops based at x gives us just enough control to build certain abelian covers separating subgroups of surface groups.

In developing a new property of groups—for example, RF, LERF, LR, or RFRS—it is imperative to understand how it behaves under various natural group operations. In Section 3, we show that the relationship $H \leq_{RS} G$ is preserved under several natural group operations:

Theorem C. *Suppose that $H_1 \leq_{RS} G_1, H_2 \leq_{RS} G_2, K_1 \leq G_1$, and $\varphi : G \rightarrow G_1$ is surjective. Then $H_1 \cap K_1 \leq_{RS} K_1, H_1 \times H_2 \leq_{RS} G_1 \times G_2$, and $\varphi^{-1}(H_1) \leq_{RS} G$. If, in addition, G_1 is LVRSS, then K_1 is LVRSS, that is, LVRSS behaves well under subgroups.*

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2. DEFINITIONS AND NOTATION

Here we collect some definitions and notation we will use throughout the rest of this paper.

Given a group G , the normal subgroup $G_r^{(1)}$ —which we will refer to as the **radical commutator subgroup** of G —is the first term in the **rational derived series**, which is defined inductively by

$$G_r^{(n+1)} := (G_r^{(n)})_r^{(1)}$$

For some results in low-dimensional topology involving the rational derived series, see [4], [5], and [9].

Given a group G , a **tower of subgroups** $\{H_i\}_{i \geq 0}$ is a collection of subgroups $H_i \leq G$ such that $H_{i+1} \leq H_i$ for all $i \geq 0$. The **limit** of a tower $\{H_i\}$ in G is

$$\lim_i H_i = \bigcap_i H_i \leq G$$

A tower $\{H_i\}$ is a **subnormal tower** if $H_{i+1} \triangleleft H_i$ for all i , and it is a **normal tower** if, in fact, $H_i \triangleleft G$ for all i .

2.1. Residual properties. Let P be a property of groups. A group G is **residually P** if for every nontrivial element $g \in G$ there is a homomorphism $\varphi : G \rightarrow F_P$ where F_P has property P and $\varphi(g) \neq 1$. Thus, a group G is **residually finite** if, for every nonidentity element $g \in G$, there exists a homomorphism $\varphi : G \rightarrow F$ onto a finite group such that $\varphi(g) \neq 1$; this definition is easily seen to be equivalent to the one given previously. Similarly, a group G is **residually free** if, for every nonidentity element $g \in G$, there exists a homomorphism $\varphi : G \rightarrow F$ to a nonabelian free group such that $\varphi(g) \neq 1$, and G is called **fully residually free** (or a **limit group**) if, for any finite collection of nontrivial elements $g_1, \dots, g_n \in G$, there is a homomorphism $\varphi : G \rightarrow F$ to a nonabelian free group such that $\varphi(g_i) \neq 1$ for all $1 \leq i \leq n$.

Let $H \leq G$. We will borrow some notation from [11]: if $H \leq G$, then define $H^* \leq G$ to be

$$H^* := \bigcap_{H \leq K \leq_f G} K$$

In general, we clearly have $H \leq H^*$, and H is separable in G if $H = H^*$.

2.2. Virtual and locally virtual properties. Let G be a group, and let P be a property of groups. G is said to be **virtually P** if G contains a finite-index subgroup $K \leq_f G$ such that K has property P . For example, G is **virtually RFRS** if there is a $K \leq_f G$ such that K is RFRS.

Given a property P of subgroups (denoted by $H \leq_P G$)—for example, H could finite-index in G ($H \leq_f G$) or RS-separable in G ($H \leq_{RS} G$)— G is said to be **locally virtually P** (or **LVP**) if, for every finitely generated subgroup $H \leq G$, there exists a finite-index subgroup $K \leq_f G$ containing H such that $H \leq_P K$.

Another local virtual property that has garnered attention recently involves retractions in groups. Let $H \leq G$. We say that G **retracts onto H** (and we call H a **retract** of G) if there exists an endomorphism $\varphi : G \rightarrow G$ such that $\varphi(G) = H$ and $\varphi|_H \equiv \text{id}_H$. We will often abuse notation and

write the retraction as $\varphi : G \rightarrow H$ without emphasizing that $H \leq G$. Similarly, we say that G **virtually retracts onto** (and we call H a **virtual retract** of G) if there is a finite-index subgroup $K \leq_f G$ and a retraction $\varphi : K \rightarrow H$. A group G admits **locally virtual retractions** (or **local retractions**, or **LR**) if every finitely generated subgroup of G is a virtual retract of G . A group G admits **virtual retractions over \mathbb{Z}** (or **local retractions over \mathbb{Z}** , or **LR over \mathbb{Z}**) if every infinite cyclic subgroup of G is a virtual retract. These notions were introduced and explored in [12] and [11].

2.3. Some new group theoretical notions. We begin this subsection with a natural generalization of Agol's RFRS condition as defined in Section 2.1:

Definition 2.1. A subgroup $H \leq G$ is **RS-separable (rationally solvably separable)**, which we denote by $H \leq_{RS} G$, if there is a descending tower of subgroups $\cdots \triangleleft_f G_2 \triangleleft_f G_1 \triangleleft_f G_0 := G$ whose limit is H and

$$(G_i)_r^{(1)} \leq G_{i+1}$$

for all $i \geq 0$. The collection $\{G_i\}$ is called an **RS tower over H** .

Thus, a group G is RFRS if $1 \leq_{RS} G$. And, if $H \leq_{RS} G$ with $\{G_i\}$ an RS tower over H , then since $G_i \leq_f G$ with $\bigcap G_i = H$, clearly

$$\bigcap G_i = H \leq H^* \leq \bigcap G_i$$

and thus $H = H^*$, that is, H is separable in G . We also note that $G \leq_{RS} G$.

In this paper, we focus on groups whose subgroups all have torsion-free abelianization, and thus we may use the following proposition to simplify our arguments:

Proposition 2.1. *Let $H \leq G$, and suppose that $H/H^{(1)}$ is torsion-free. Then $H^{(1)} = H_r^{(1)}$.*

Proof. Clearly $H^{(1)} \subset H_r^{(1)}$, so it is enough to show that $H^{(1)} \supset H_r^{(1)}$. Suppose this fails, that is, there is an element $g \in H_r^{(1)} \setminus H^{(1)}$. So, $g^d \in H^{(1)}$ for some $|d| \geq 2$ but $g \notin H^{(1)}$. Then $1 \neq \bar{g} \in H/H^{(1)}$, but $1 = \overline{g^d} \in H/H^{(1)}$, contradicting the fact that $H/H^{(1)}$ is torsion-free. \square

Since both LERF and RFRS are natural extensions of residual finiteness, we would like to combine them, naturally, into a single property. To do this, we recall that for Agol's purposes in [2], it is more important for a group to contain a finite-index subgroup that is RFRS rather than require that the group itself is RFRS. In other words, Agol is interested in those groups G that are **virtually RFRS**, that is, there exists $K \leq_f G$ such that $1 \leq_{RS} K$. Thus, in order to generalize this in a natural way, we borrow some ideas from Section 2.2; for one, we will say that a subgroup $H \leq G$ is **virtually RS separable** (denoted $H \leq_{VRS} G$) if there exists a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} K$. For example, all finite-index subgroups $H \leq_f G$ are virtually RS separable because $H \leq_{RS} H \leq_f G$. The main focus of this paper is an investigation of the following property:

Definition 2.2. A group G is **LVRSS (locally virtually RS-separable)** if, for every finitely generated subgroup $H \leq G$, there exists a finite-index subgroup $K \leq G$ such that $H \leq_{RS} K$.

We hasten to point out that this is not a trivial definition. All finite groups are LVRSS because, if G is a finite group and H is a (finitely generated) subgroup of G , then $H \leq_f G$ and $H \leq_{RS} H$. It's not hard to see that all finitely generated abelian groups are LVRSS as well.

Although LVRSS clearly implies virtually RFRS, it is not completely obvious that LVRSS implies LERF, so we provide justification for this fact.

Proposition 2.2. *Suppose that G is LVRSS. Then G is LERF.*

Proof. Let $H \leq G$ be a finitely generated subgroup of G . We must show that H is separable in G . Since G is LVRSS, there is a finite-index subgroup $K \leq_f G$ such that $H \leq_{RS} K$, and therefore, by the above observation, H is separable in K . But this implies that H is separable in G because every

subgroup of finite index in K also has finite index in G , so the intersection of all subgroups of finite index in K containing H contains the intersection of all subgroups of finite index in G containing H . Thus, H is separable in G . \square

This new property of groups, LVRSS, fits into the existing literature in a few ways. On the one hand, it is a very strong form of residual finiteness. On the other hand, since it implies both LERF and virtually RFRS, it may be thought of as a certain combination of those two properties, each of which imply residual finiteness as well. To justify the legitimacy of these new definitions, we prove in Section 3 that RS separability is well-behaved under some natural operations on groups, and we compare the behavior to that of RFRS and subgroup separability. Furthermore, we show in sections 4 and 5 that free groups and closed (orientable) surface groups are LVRSS, and so LVRSS shares the property of generalizing “freeness” with residual finiteness, LERF, and (virtually) RFRS.

3. BEHAVIOR OF RS SEPARABILITY

3.1. Intersection and subgroups. We start with a basic observation, which, since we will use it repeatedly throughout this paper, we record as a lemma:

Lemma 3.1. *Suppose that $G_1, G_2 \leq G$ such that $(G_1)_r^{(1)} \leq G_2$, and let $K \leq G$. Then*

$$(G_1 \cap K)_r^{(1)} \leq G_2 \cap K$$

Proof. If $x \in (G_1 \cap K)_r^{(1)}$, then $x \in G_1 \cap K \leq K$, and $x^d \in (G_1 \cap K)^{(1)} \leq G_1^{(1)}$ for some $d \neq 0$. Thus $x \in (G_1)_r^{(1)} \cap K \leq G_2 \cap K$. \square

LERF is well known to be preserved by subgroups; see, for example, Lemma 1.1 in [14]. It is easy to see that RFRS is preserved by subgroups as well; see the remarks following Definition 2.1 in [2]. Now we use Lemma 3.1 to show that, perhaps as expected, the same is true of RS separability:

Proposition 3.2. *Suppose that $H \leq_{RS} G$ and $K \leq G$. Then $H \cap K \leq_{RS} K$.*

Proof. Let $\{G_i\}$ be an RS tower over H in G , so $(G_i)_r^{(1)} \leq G_{i+1}$ for $i \geq 0$ and

$$\cdots \triangleleft_f G_2 \triangleleft_f G_1 \triangleleft_f G_0 := G$$

such that $H = \bigcap G_i$. Set $K_i := G_i \cap K$. Then $\bigcap K_i = H \cap K$, and Lemma 3.1 yields $K_{i+1} \geq (K_i)_r^{(1)}$. Therefore $\{K_i\}$ is an RS tower over $H \cap K$, so $H \cap K \leq_{RS} K$. \square

By setting $H = 1$ in Proposition 3.2, we recover the observation that RFRS passes to subgroups. A fortiori, LVRSS passes to subgroups as well; this is the content of the following:

Corollary 3.3. *Suppose that G is LVRSS, and let $H \leq G$. Then H is LVRSS as well.*

Proof. Let $L \leq H$ be a finitely generated subgroup of H . Then L is a finitely generated subgroup of G , and thus, since G is LVRSS, there exists a finite-index subgroup $K \leq_f G$ such that $L \leq_{RS} K$. But $H \cap K \leq_f H$, and $L \leq H \cap K$, so by Proposition 3.2,

$$L = L \cap (K \cap H) \leq_{RS} K \cap H \leq_f H,$$

that is, L is RS separable in $K \cap H \leq_f H$, and thus $L \leq_{VRS} H$. Since L was an arbitrary finitely generated subgroup of H , it follows that H is LVRSS. \square

As another application of Lemma 3.1, we prove an element-wise characterization of RS separability. This is analogous to the fact that a subgroup H is separable in G if and only if, for every $g \in G \setminus H$ there exists a finite-index subgroup $K \leq_f G$ such that $H \leq K$ and $g \in G \setminus K$.

Proposition 3.4. *Let G be finitely generated. Then $H \leq_{RS} G$ if and only if the following condition holds:*

(*) For every $g \in G \setminus H$, there exists an $n = n_g \geq 0$ and a finite subnormal tower $G = G_0 \triangleright_f G_1 \triangleright_f \cdots \triangleright_f G_n$ such that $g \notin G_n \leq_f G$, $H \leq G_n$, and $G_{i+1} \geq (G_i)_r^{(1)}$ for all $0 \leq i \leq n-1$.

Proof. First suppose that $H \leq_{RS} G$, and let G_i be an RS tower over H with $\cap G_i = H$. Then if $g \in G \setminus H$, there is an n such that $g \notin G_n$, so $G_0 \geq \cdots \geq G_n$ is the desired tower.

Now suppose that (*) holds, and list the elements of $G \setminus H = \{g_1, g_2, \dots\}$. By (*), there is a subnormal tower $G = G_0 \geq G_1 \geq \cdots \geq G_{n_1}$ that is an RS tower over G_{n_1} such that $g_1 \notin G_{n_1}$ and $H \leq G_{n_1}$. Let i be minimal such that $g_i \in G_{n_1} \setminus H$. By (*), there is a tower $G = G'_0 \geq G'_1 \geq \cdots \geq G'_{n_2}$ such that $g \notin G'_{n_2}$ and $H \leq G'_{n_2}$. Then

$$G_0 \geq G_1 \geq \cdots \geq G_n = G_n \cap G'_0 \geq G_n \cap G'_1 \geq \cdots \geq G_n \cap G'_{n_2}$$

is a subnormal RS tower over $G_{n_1} \cap G'_{n_2}$ because $G'_{i+1} \geq (G'_i)_r^{(1)}$ implies $G_n \cap G'_{i+1} \geq (G_n \cap G'_i)_r^{(1)}$ by Lemma 3.1. Furthermore, $g_i \notin G_{n_1} \cap G'_{n_2}$, so continuing this process with the first element $g_j \in (G_{n_1} \cap G'_{n_2}) \setminus H$, we get an RS tower G_i over H such that $\cap G_i = H$ as needed (because $g_k \notin G_{n_k}$). \square

3.2. Direct product. One might hope that a property such as LVRSS would be preserved by such a natural group operation as the direct product; that is, one might expect that if G_1 and G_2 are LVRSS, then $G_1 \times G_2$ is LVRSS as well. Indeed, other related properties—such as residual finiteness, RFRS, LR, and LR over \mathbb{Z} —are preserved by direct product. However, things are not quite so simple for LERF (and, consequently, for LVRSS) because, for example, although the free group of rank two F_2 is LVRSS (see Section 4), $F_2 \times F_2$ is not even LERF, and hence by Proposition 2.2, $F_2 \times F_2$ is not LVRSS. However, we can prove that RS separable subgroups are preserved under direct product:

Proposition 3.5. *If $H_1 \leq_{RS} G_1$ and $H_2 \leq_{RS} G_2$, then $H_1 \times H_2 \leq_{RS} G_1 \times G_2$.*

Proof. Let $\cdots \triangleleft_f G_{1,1} \triangleleft_f G_{1,0} := G_1$ be an RS tower for $H_1 \leq G_1$, and let $\cdots \triangleleft_f G_{2,1} \triangleleft_f G_{2,0} := G_2$ be an RS tower for $H_2 \leq G_2$. As noted in [2],

$$(G_{1,i} \times G_{2,i})_r^{(1)} = (G_{1,i})_r^{(1)} \times (G_{2,i})_r^{(1)} \leq G_{1,i+1} \times G_{2,i+1}$$

Furthermore,

$$(3.1) \quad \bigcap (G_{1,i} \times G_{2,i}) = \cap G_{1,i} \times \cap G_{2,i} = H_1 \times H_2$$

Thus, $G_{1,i} \times G_{2,i}$ is an RS tower over $H_1 \times H_2$. \square

Note that Equation 3.1 is essentially the proof that if H_1 is separable in G_1 and H_2 is separable in G_2 , then $H_1 \times H_2$ is separable in $G_1 \times G_2$.

3.3. Preimages of homomorphisms. We would like to know to what extent RS separability is preserved under preimages of homomorphisms. We begin with an auxiliary lemma that says that the commutator radical is preserved by homomorphisms as well as their inverse images.

Lemma 3.6. *Suppose that $\varphi : G \rightarrow H$ is a homomorphism. Then $\varphi(G_r^{(1)}) \leq H_r^{(1)}$, so $G_r^{(1)} \leq \varphi^{-1}(H_r^{(1)}) \leq G$.*

Proof. Suppose that $x \in G_r^{(1)}$, so $x^d = \prod_i [y_i, z_i] \in G^{(1)}$ for some $d \neq 0$ and some $y_i, z_i \in G$. Now compute:

$$(\varphi(x))^d = \varphi(x^d) = \varphi\left(\prod_i [y_i, z_i]\right) = \prod_i [\varphi(y_i), \varphi(z_i)]$$

Thus, $(\varphi(x))^d \in H^{(1)}$, and hence $\varphi(x) \in H_r^{(1)}$ as needed. The second claim follows since $G_r^{(1)} \leq \varphi^{-1}(\varphi(G_r^{(1)}))$. \square

An elementary corollary of this lemma is that RS-separable subgroups are preserved under inverse images:

Corollary 3.7. *Suppose that $\varphi : G \rightarrow H$ is a homomorphism, and $K \leq_{RS} H$. Then $\varphi^{-1}(K) \leq_{RS} \varphi^{-1}(H)$. In particular, preimages of RS towers over $K \leq H$ are RS towers over $\varphi^{-1}(K) \leq \varphi^{-1}(H)$.*

Proof. Let $\cdots \triangleleft_f H_1 \triangleleft_f H_0 := H$ be an RS tower over K , and set $G_i := \varphi^{-1}(H_i)$. Since pre-images preserve intersections, we have

$$\bigcap G_i = \bigcap \varphi^{-1}(H_i) = \varphi^{-1}(\bigcap H_i) = \varphi^{-1}(K)$$

Furthermore, by applying Lemma 3.6 to $\varphi|_{\varphi^{-1}(H_i)} : \varphi^{-1}(H_i) \rightarrow H_i$, we find

$$G_{i+1} = \varphi^{-1}(H_{i+1}) \geq \varphi^{-1}((H_i)_r^{(1)}) \geq (\varphi^{-1}(H_i))_r^{(1)} = (G_i)_r^{(1)}$$

and therefore $\cdots \triangleleft_f \varphi^{-1}(H_1) \triangleleft_f \varphi^{-1}(H_0) = \varphi^{-1}(H)$ is an RS tower over $\varphi^{-1}(K)$. \square

It is a fact known to Agol (and communicated to the author by Henry Wilton) that all residually free groups are RFRS; however, no proof exists in the literature. As another application of Lemma 3.6, we provide a proof here.

Proposition 3.8. *Finitely generated residually free groups are RFRS.*

Proof. Suppose that G is residually free; note that all subgroups of G are residually free as well. List the elements of $G = \{1 = g_0, g_1, g_2, \dots\}$. Since $G_0 := G$ is residually free, there is a homomorphism $\varphi_0 : G_0 \rightarrow F_0$ (where F_0 is free) such that $\varphi_0(g_0) \neq 1$. Free groups are RFRS (see [2]), so there is a RFRS tower $\cdots \triangleleft_f F_{0,1} \triangleleft_f F_{0,0} := F_0$. Since $\bigcap F_{0,i} = \{1\}$, there is a minimal r_0 such that $\varphi_0(g_0) \in F_{0,r_0-1} \setminus F_{0,r_0}$. Now set $G_{0,i} := \varphi^{-1}(F_{0,i})$ for $1 \leq i \leq r_0$, and set $G_1 := G_{0,r_0}$. By Lemma 3.6, $G_{0,i+1} \geq \varphi_0^{-1}((F_{0,i})_r^{(1)}) \geq (G_{0,i})_r^{(1)}$. For $k \geq 1$, apply this procedure to the residually free group G_k (and the next $g_i \in G_k$) to get $\varphi_k : G_k \rightarrow F_k$. Then $g_k \notin G_{k+1}$ for each k , so $\bigcap G_k = \{1\}$. \square

4. FREE GROUPS ARE LVRSS

In Section 2, we provided some examples of LVRSS groups. The main result of this section is that free nonabelian groups are also LVRSS (Corollary 4.9). The main ingredients in the proof are Marshall Hall's Theorem [8], a theorem on bases of subgroups of free groups due to Federer and Jónsson [6], and some techniques inspired by the theory of local retractions [11].

4.1. Free groups and graphs. Let X be a set. We will denote by F_X the **free group** on X , which means that there is a map $\iota : X \rightarrow F_X$ such that, for any set map $f : X \rightarrow G$ to any other group G , there exists a unique group homomorphism $\varphi : F_X \rightarrow G$ that makes the following diagram commute:

$$\begin{array}{ccc} X & \longrightarrow & F_X \\ & \searrow f & \downarrow \varphi \\ & & G \end{array}$$

It is a standard fact that any element of $A * B$ has a unique so-called “reduced form,” and, as a consequence, $A, B \hookrightarrow A * B$. We will require a notion of reduced form just in free groups, and for this, we follow subsection 1 of section 1 of [13]. Let F_X be a free group with basis X . A word in X is called **reduced** if contains no part xx^{-1} for $x \in X^{\pm 1}$. Each equivalence class of words contains a unique reduced word. We will let $L_X(w)$ denote the X -length of a reduced word $w \in F$ over $X^{\pm 1}$ —that is, $L_X(w)$ is the length of the reduced word representing w , and we will write $w = w(1)w(2) \cdots w(n)$ with each $w(i) \in X^{\pm 1}$. We will sometimes use the notation F_r to denote F_X with $|X| = r$.

We observe that, using the universal property of free groups, it is not difficult to show that $F_{X \sqcup Y} \cong F_X * F_Y$. However, using group presentations is even easier, since each of these is presented $\langle X, Y \mid - \rangle$. For this reason, we may think of $F_X, F_Y \leq F_{X \sqcup Y}$. As the following elementary example shows, retracts occur naturally in free groups:

Proposition 4.1. *There exists a canonical retraction $F_{X \sqcup Y} \rightarrow F_X$.*

Proof. Let $i : X \rightarrow F_X$ be the set inclusion given by the definition of F_X and define a set map $f : X \sqcup Y \rightarrow F_X$ by

$$f(a) = \begin{cases} i(a) & a \in X \\ 1 & a \in Y \end{cases}$$

By the universal property of free groups, there is a unique homomorphism $F_{X \sqcup Y} \rightarrow F_X$ corresponding to f . The induced map $F_{X \sqcup Y} \rightarrow F_X$ is clearly the identity on $F_X \leq F_{X \sqcup Y}$. \square

4.2. Marshall Hall's Theorem and retractions. In this subsection, we recall the well-known theorem of Marshall Hall Jr. on subgroups of free groups. We begin this subsection with a useful topological criterion (due to Scott [14]) for testing whether a subgroup is separable:

Proposition 4.2. *(Scott's Criterion, [14]) Let X be a Hausdorff topological space and $G = \pi_1(X, x)$. Let $(X', x') \rightarrow (X, x)$ be a covering and $H = \pi_1(X', x')$. Then H is separable in G if and only if for any compact subset $\Delta \subset X'$, there exists an intermediate finite-sheeted cover $X' \rightarrow \hat{X} \rightarrow X$ such that $X' \rightarrow \hat{X}$ embeds Δ into \hat{X} .*

Proof. See Lemma 1.4 in [14]. \square

Scott's Criterion may be applied to covering spaces of graphs, as in Stallings' proof that free groups are LERF (Marshall Hall's Theorem), and in Section 5, we will apply Scott's Criterion to covering spaces of closed surfaces. We note that Stallings' proof tells us that $H \leq F_r$ is a free factor of a finite-index subgroup of F_r . By Proposition 4.1, this implies that free groups admit local retractions: for any finitely generated $H \leq F_r$, there is a finite-index subgroup $K \leq_f F_r$ such that $K = H * F'$ for some $F' \leq F_r$, and the natural quotient $\varphi : K \rightarrow H$ is a retraction.

4.3. Orderings and free bases. The goal of this subsection is to make sense of a certain result (recorded as Lemma 4.4 below) on free bases of subgroups of free groups due to Federer and Jónsson [6]. We first recall some definitions of orderings on sets.

A binary relation \leq is a **total order** on a set X if the following three conditions are satisfied for all $a, b, c \in X$:

- (1) if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetry);
- (2) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity); and
- (3) either $a \leq b$ or $b \leq a$ (totality).

To any total order \leq is associated a **strict total order** $<$, which may be defined as $a < b$ if and only if $a \leq b$ and $a \neq b$. A total order \leq on a set X is a **well-ordering** if every nonempty subset of X has a least element under \leq .

If now F is a group and $G \leq F$ with well-ordering \leq and strict total ordering $<$ on a subgroup $G \leq F$ of a group F , define for any element $g \in G$:

$$G_g := \langle \{h \in G \mid h < g\} \rangle \leq G$$

That is, G_g is the subgroup of G generated by all elements of G that are strictly less than g under the well-ordering \leq . For example, if G is generated by X , and $x < g$ for all $x \in X$, then $G_g = G$. As another example involving G_g , we state a basic fact which we will use below:

Proposition 4.3. *If $H < G$ is a proper subgroup of G and g is the least element in the set $G \setminus H$, then $G_g \leq H$. In particular, $g \notin G_g$.*

Proof. By the choice of g , if $h \in G$ is such that $h < g$, then $h \in H$. Thus $G_g \leq H$. The second assertion clearly follows from the first. \square

Recall from Section 4 that, for an element $g \in G$, $L_X(g)$ is the length of the unique reduced word in $X^{\pm 1}$ representing g . The definition of G_g is central to the following result of Federer and Jónsson (see Theorem 4.2 in [6], or Proposition 2.10 in [13]), and we will use this to show that free groups are locally virtually RS separable:

Lemma 4.4. *Let G be a subgroup of the free group F , and let G be well ordered by any relation $<$ such that $g < h$ implies $L_X(g) \leq L_X(h)$. Then the set*

$$A = A_G = \{g \in G \mid g \notin G_g\}$$

is a free basis for G .

In the situation of Lemma 4.4, it is not difficult to see that $\langle A \rangle = G$: if $\langle A \rangle$ is a proper subgroup of G and g is the least element of $G \setminus \langle A \rangle$, then Proposition 4.3 implies that $g \notin G_g$, so $g \in A$, a contradiction. Hence, the content of the lemma is that A is a *free* basis for G ; however, this set clearly generates its associated subgroup in an arbitrary (that is, not necessarily free) group. We record this observation as a proposition to which we will refer in Section 5:

Proposition 4.5. *Let $(G, <)$ be a group G with a well-ordering $<$, and let $H \leq G$. For any $g \in H$, define*

$$H_g = \langle \{h \in H : h < g\} \rangle$$

Then the set

$$A_H = \{h \in H : h \notin H_h\}$$

generates H .

We list a basic consequence of Lemma 4.4 for convenience.

Corollary 4.6. *Let G be a subgroup of the free group F , and let G be well ordered by any relation $<$ such that $g < h$ implies $L_X(g) \leq L_X(h)$, and let $H < G$ be a proper subgroup of G . Then the least element g in $G \setminus H$ is in the basis $A = \{g \in G \mid g \notin G_g\}$ of G .*

Proof. This is an immediate consequence of Proposition 4.3 and Lemma 4.4. \square

We will apply this result by building a well-ordering on $F = F_X$ with the desired property; this is the content of the following basic lemma:

Lemma 4.7. *Any strict total order $<$ on the finite set X may be extended to a well-ordering $<$ on the free group F_X such that $u < v$ implies $L_X(u) \leq L_X(v)$ and such that any $x \in X$ is less than any nontrivial element in $F_X \setminus X$.*

Proof. Relabel elements of X so that $x_1 < x_2 < \dots < x_n \in X$. Extend this to the symmetric set X^{\pm} :

$$x_1 < \dots < x_n < x_1^{-1} < \dots < x_n^{-1}$$

Define the relation \leq on reduced words in $u, v \in F_X$ by: $u \leq v$ if and only if $L_X(u) < L_X(v)$ or, if $L_X(u) = L_X(v)$, then $u(i) < v(i)$ for the minimal i where u and v differ, or $u = v$.

First, we argue that this is a total order. If $u \leq w$ and $w \leq u$, then $L_X(u) = L_X(w)$ and $u(i) = w(i)$ for all i , so $u = w$. If $u \leq w$ and $w \leq v$, then we may assume $L_X(u) = L_X(v)$, $u \neq w$, $w \neq v$. But u and w are equal words through the i -th letter (for some i), and w and v are equal words through the j -th letter for some j . So if $i < j$, then u and v are equal through $u(i) < w(i) = v(i)$, hence $u \leq v$. Similarly, if $i > j$ or $i = j$, then $u \leq v$, as needed. To complete the proof that \leq is a total order, suppose $L_X(u) = L_X(w)$. If $u \neq w$, then u and w differ at some letter, hence $u \leq w$ or $w \leq u$.

To show that \leq is a well-ordering, we must show that every nonempty subset $Y \subseteq F_X$ has a least element. But this follows from the definition of \leq and the fact that there are only finitely many elements in F_X of a given (positive, finite) word length.

The last assertion is obvious by the definition of \leq since

$$\{g \in F_X : L_X(g) = 1\} = X^{\pm 1}.$$

□

4.4. RS separating subgroups of free groups. Now we will apply the results of Section 4.3 to find certain elements of free bases of subgroups of a free group, and we will use these elements to construct an RS tower.

Theorem 4.8. *Suppose that X, Y are finite sets. Then $F_X \leq_{RS} F_X * F_Y$.*

Note: It is not enough to consider the composition of the natural maps $F_X * F_Y \rightarrow F_Y \rightarrow \mathbb{Z}^{|Y|}$ because this kills all conjugates of F_X , so intersecting kernels would yield at least the normal closure of F_X in $F_X * F_Y$, which is much bigger than F_X . A fortiori, for the same reason it is not enough to pull back a RFRS tower via $F_X * F_Y \rightarrow F_Y$; its pre-image would intersect in the kernel of the homomorphism, a normal subgroup of $F_X * F_Y$.

Proof. We may assume that X and Y are nonempty. Set $F := F_X * F_Y$, and fix a strict total ordering $<$ on $X \sqcup Y$ such that elements of X^{\pm} are less than those of Y^{\pm} . By Lemma 4.7, this extends to a well-ordering \leq on $F_{X \sqcup Y} \cong F_X * F_Y$ such that $u < v$ implies $L_{X \sqcup Y}(u) \leq L_{X \sqcup Y}(v)$ and such that elements of $X \sqcup Y$ are less than nontrivial elements in $F \setminus (X \sqcup Y)$. Throughout this proof, given a subgroup $F_i \leq F$, let A_i denote the basis of F_i resulting from Lemma 4.4, where word length $L_{X \sqcup Y}(w)$ is calculated over $X \sqcup Y$ in F , and the well-ordering on $F_i \leq F$ is the restriction of the well-ordering \leq on F . Note that if any such subgroup F_i contains X , then the elements of X will be the least elements in F_i , and therefore in this situation, $X \subset A_i$.

Set $F_0 = F$, and let g_0 be a least element of the subset $F_0 \setminus F_X \subset F_0$ of the well-ordered set F_0 . Then $g_0 \in A_0$ by Corollary 4.6, and thus $X \sqcup \{g_0\} \subset A_0$. Let F_1 be the kernel of the natural map $F_0 \rightarrow \langle g_0 \rangle \rightarrow \langle g_0 \rangle / \langle g_0^2 \rangle$; in particular, $X \subset F_X \subset F_1$. Thus we have an RS tower $F_0 > F_1$ with $F_1 \geq F_X$.

Assume that we have built an RS tower $F_0 > F_1 > \dots > F_i$ with $F_i \geq F_X$. Let g_i be a least element in $F_i \setminus F_X$, so $g_i \in A_i$ by Corollary 4.6, and let F_{i+1} be the kernel of $F_i \rightarrow \langle g_i \rangle \rightarrow \langle g_i \rangle / \langle g_i^2 \rangle$. Repeating in this fashion, we get a tower $F_0 > F_1 > F_2 > \dots$ over F_X . By the construction of the tower, the function

$$f(n) = \min\{L_{X \sqcup Y}(u) : u \in F_n \setminus F_X\}$$

is non-decreasing, and $g_i \notin F_{i+1}$ for each $i \geq 0$. Thus, since there are only finitely many reduced words in F_0 of a given finite length, for any $N \geq 1$ there is a number m_N such that $f(m_N) > N$, so we have $\cap F_i = F_X$. Additionally, since the abelianization of each F_i is torsion-free, the tower $\{F_i\}$ is an RS tower because $F_i/F_{i+1} \cong \mathbb{Z}/2\mathbb{Z}$ for each i . Thus, we conclude that $F_X \leq_{RS} F_X * F_Y$. □

Corollary 4.9. *If H is a finitely generated subgroup of a finitely generated free group F , then $H \leq_{VRS} G$. In other words, finitely generated free groups are LVRSS.*

Proof. By Hall's theorem [8], there is a finite-index subgroup $K \leq_f F$ containing H as a free factor, $K = H * F'$. Apply Theorem 4.8 to $H \leq K$. □

Thus, finitely generated free groups are LVRSS. In some sense, this result is as strong as possible because not all finitely generated subgroups of free groups are RS-separable. To see this, let $H \leq G_0 = G$ (where G is a free group) with H surjecting G^{ab} , and suppose that G_1 is the first term in a potential RS tower over H :

$$H \leq G_1 \triangleleft_f G_0 = G$$

with G_0/G_1 abelian. Then we have the following diagram:

$$\begin{array}{ccc}
 H & \xrightarrow{i} & G_0 \\
 & \searrow & \downarrow \\
 & & G_0/[G_0, G_0] \longrightarrow G_0/G_1
 \end{array}$$

If $\ker(p) = G_1 < G_0$ were a proper subgroup, then $H \cap G_1 = \ker(p \circ i) < H$ would be a proper subgroup of H , contradicting the assumption $H \leq G_1$. Thus, $G_1 = G_0$. Therefore, a proper subgroup that surjects $F/[F, F]$ is not RS-separable; we record this observation in the following:

Proposition 4.10. *Let F be a nonabelian free group. If $H < F$ is a proper subgroup such that the composition $H \hookrightarrow F \rightarrow F/[F, F]$ is surjective, then H is not RS separable in F .*

In Section 5, we will show that closed, orientable surface groups are also locally virtually RS-separable.

5. SURFACE GROUPS ARE LVRSS

In this section, we prove that all finitely generated subgroups of an orientable surface group are virtually RS separable (Theorem 5.13). This, along with Corollary 4.9 implies that all orientable surfaces of finite type have LVRSS fundamental groups.

This section is organized as follows: Section 5.1 contains some topological lemmas that allow us to build cyclic covers to which some curves lift but others do not; Section 5.2 details the way in which we find the nice finite-index subgroup G' and is analogous to Section 4.2 in Section 4; in Section 5.3, we record some facts about the geometry of hyperbolic surfaces and geodesic loops, including the fact that the collection A_S of geodesic loops based at x that are not products of shorter loops consists of simple, minimally intersecting loops, and in fact generates $\pi_1(S, x)$. Finally, in Section 5.4, we prove the main theorem.

5.1. Surface groups and topology. In this subsection, we accomplish two things: first, we give a sufficient condition for constructing a double cover of a surface that kills a particular non-separating curve. Second, we construct a double cover to which a particular separating curve lifts to a non-separating curve. These results are used in the proof of Theorem 5.13 in Section 5.4.

Let $S = S_{g,n}$ be a compact, orientable, connected surface of genus g with n boundary components. Let $\gamma : [0, 1] \rightarrow S$ be a **closed curve**, that is, γ is a smooth map such that $\gamma(0) = \gamma(1)$. We will often abuse notation and let $\gamma \subset S$ denote the image of the curve γ . γ is a **simple closed curve** if γ has no self-intersections (that is, $\gamma(x) \neq \gamma(y)$ for all $0 \leq x < y \leq 1$). If γ is a simple closed curve, we call γ a **non-separating curve** if $S \setminus \gamma$ is connected, and otherwise call γ a **separating curve**. A **multicurve** is a collection of disjoint simple closed curves.

Given a closed curve γ , the homology class (with \mathbb{Z} coefficients) of γ is denoted $\bar{\gamma}$. A closed curve γ that is not null-homotopic is called **primitive** if it is simple and $\bar{\gamma}$ is primitive in $H_1(S)$. So, if γ is primitive, then either γ is non-separating or γ is boundary parallel and $n > 1$. If α, β are closed curves on S , then $i(\alpha, \beta)$ is the geometric intersection number of α and β . For multicurves $\mu, \nu \subset S$, we define intersection additively:

$$i(\mu, \nu) = \sum_{\alpha \in \mu, \beta \in \nu} i(\alpha, \beta)$$

We also note that every non-separating curve γ is primitive. This is because there is a dual curve δ (which is easy to find after cutting along γ , for example) such that $T = N(\gamma \cup \delta) \cong S_{1,1}$ with $H_1(T) = \langle \bar{\gamma} \rangle \oplus \langle \bar{\delta} \rangle$. And, in this case $H_1(S) = H_1(T) \oplus H_1(S \setminus T)$, so γ is primitive in $H_1(S)$ since it is obviously primitive in $H_1(T)$. More generally:

Lemma 5.1. *Let $K \hookrightarrow S$ be an essential subsurface, and let $\gamma \subset S$ be a non-separating simple closed curve such that $\gamma \subset S \setminus K$, or $i(\gamma, \partial K) = 2$ and γ intersects two distinct components of ∂K . Then there is a map $f : H_1(S) \rightarrow \mathbb{Z}/2\mathbb{Z}$ such that the induced $\pi_1(K) \rightarrow H_1(S) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is trivial but $f(\overline{\gamma})$ is nonzero. In other words, there is a double cover $S' \rightarrow S$ to which K lifts but γ does not.*

Proof. If $\gamma \subset S \setminus K$, then the proof follows from the above observation, with f equal to the composition of the natural projection onto $H_1(T)$, followed by projection onto $\langle \overline{\gamma} \rangle$, and finally by the projection onto $\mathbb{Z}/2\mathbb{Z}$; here, we may take $T \subset S \setminus K$.

Now suppose that $i(\gamma, \partial K) = 2$ and γ intersects two distinct components k_1, k_2 of ∂K . Note that since K is essential, k_1 and k_2 are primitive in $H_1(S)$, so there is a map $\phi : H_1(S) \rightarrow \mathbb{Z}$ defined by algebraic intersection number with k_1 , say. Notice that $\alpha \cdot k_1 = 0$ for any $\alpha \subset K$, so $\phi(K) = 0$. However, $\gamma \cdot k_1 = \pm 1$ (depending on the orientations), so we find the desired map $f : H_1(S) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by composing ϕ with $\langle \overline{\gamma} \rangle \rightarrow \langle \overline{\gamma} \rangle / \langle \overline{\gamma}^2 \rangle$. The last statement follows by the lifting criterion. \square

In order to apply Lemma 5.1, we may need to lift a separating curve outside of an essential subsurface to a nonseparating curve outside the lifted subsurface in a double cover:

Lemma 5.2. *Suppose that Σ is a closed surface with a π_1 -injective subsurface $K \subset \Sigma$. Let γ be a simple closed curve in a component S of $\Sigma \setminus K$ that is essential and separating on S . Then there exists a double cover $\Sigma' \rightarrow \Sigma$ where K and γ lift to K' and γ' such that γ' is not separating on its connected component of $\Sigma' \setminus K'$.*

Proof. We consider two cases depending on the number of components in ∂S . If ∂S has one component, then $S \setminus \gamma \cong S_{k,2} \sqcup S_{g-k,1}$ with $0 < k < g$ (since γ bounds neither a disk nor an annulus in S). Thus there is an essential simple closed curve δ_i in the i -th component of $S \setminus \gamma$ ($i = 1, 2$), such that neither δ_1 nor δ_2 is separating on Σ . Cut Σ along δ_1, δ_2 and glue two copies of the resulting (connected) surface appropriately to find the desired double-cover; see Figure 1.

If ∂S has more than one component, then each component of $S \setminus \gamma$ either has positive genus or has at least three boundary components, and in either case we find nontrivial (but possibly boundary parallel, in the genus 0 case) simple closed curves δ_i in the i -th component of $S \setminus \gamma$, neither of which is separating on Σ , and we cut and glue two copies of the connected surface $\Sigma \setminus (\delta_1 \cup \delta_2)$ to find the desired cover; see Figure 2. \square

Lemma 5.3. *Let (Σ, x) be a based closed surface of genus $g \geq 2$. Let $(K, x) \hookrightarrow (\Sigma, x)$ be a π_1 -injective subsurface. Let (β, x) be a simple loop based at x such that $\beta \notin \pi_1(K, x)$ and β is freely homotopic into a component S of $\Sigma \setminus K$. Then there is a tower of no more than three double covers such that K lifts to the largest cover but β does not.*

Proof. Throughout this proof, let $\gamma \subset S$ denote the simple closed curve in S to which β is freely homotopic.

First assume that γ is non-separating on S . Then applying Lemma 5.1 yields the desired double cover.

Next, assume that γ is separating but not essential on S . Then Lemma 5.2 yields a double cover to put us in the above case, so we find a tower of two double covers that achieves the goal.

Finally, assume that γ is separating and nonessential on S , that is, γ is freely homotopic to ∂S . In this case, ∂S has at least two components because, by the assumption that $\beta \notin \pi_1(K, x)$, β is not freely homotopic to the component of ∂S that it intersects. Cut along the boundary component to which it is homotopic and glue two copies to find a double cover where β lifts to a separating essential simple closed curve on S ; the picture is similar to Figure 2. This puts us in the previous case, and thus β does not lift although K does in some tower of three double covers, as claimed. \square

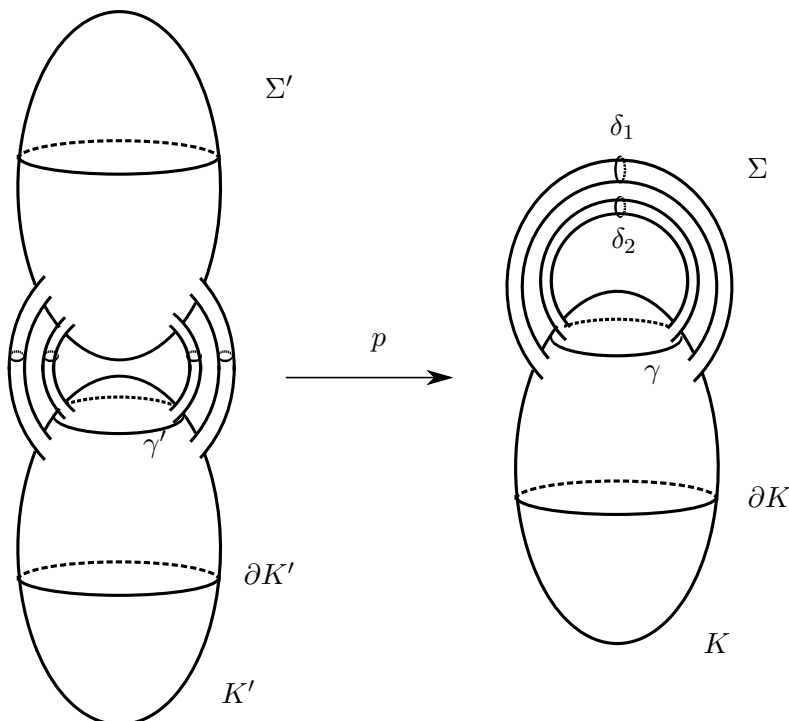


FIGURE 1. Constructing a double-cover $p : \Sigma' \rightarrow \Sigma$ where γ' is not separating in $\Sigma' \setminus K'$ and ∂K has one component.

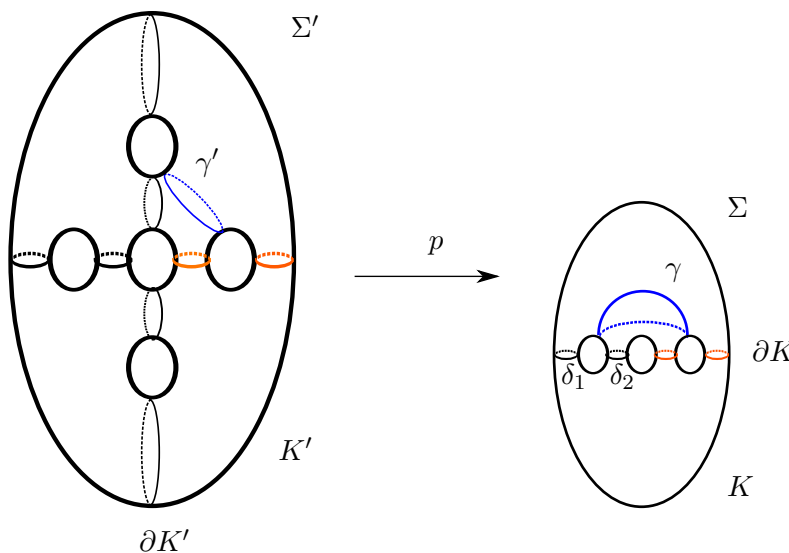


FIGURE 2. Constructing a double-cover $p : \Sigma' \rightarrow \Sigma$ where γ' is not separating in $\Sigma' \setminus K'$ and ∂K has at least two components.

5.2. **Scott's theorem.** Here we record a theorem due to Peter Scott (Theorem 3.1 in [14]):

Theorem 5.4. *Let G be the fundamental group of a closed surface. Then G is LERF.*

In fact, Theorem 2.6 from [11] tells us a bit more:

Theorem 5.5. *Suppose that G is the fundamental group of a closed surface of negative Euler characteristic. Then G is LR.*

The main idea from the proof of Theorem 5.4 that we will use again in the proof of Theorem 5.13 is the notion of a compact core. Let S be a closed surface. Let H be a finitely generated subgroup of G . There is a **compact core** $C \subset S_H$, that is, C is a compact, essential subsurface of S_H .

Theorem 5.4, along with Scott's Criterion (Proposition 4.2 in Section 4) immediately yield the following:

Proposition 5.6. *Let S be a closed, hyperbolic surface and $G = \pi_1(S, x)$. Let $H \leq G$ be finitely generated, and let $(S_H, x_H) \rightarrow (S, x)$ be the associated covering. Then for any compact subset $C \subset S_H$, there exists an intermediate finite-sheeted cover $S_H \rightarrow S' \rightarrow S$ such that $S_H \rightarrow S'$ isometrically embeds C into S' .*

In Section 4, we used Scott's Criterion with the fact that free groups are LERF (Hall's Theorem) to pass to a finite-index subgroup $F' \leq F$ where a given finitely generated $H \leq G$ appears as a free factor and we proved that free factors are RS separable in free groups. In the present section, we will use Proposition 5.6 to find a finite-sheeted cover $S' \rightarrow S$ that contains a large compact set C that carries $H \leq \pi_1(S, x)$.

5.3. Geometry of hyperbolic surfaces. The hyperbolic plane \mathbb{H}^2 is a complete, simply connected Riemannian surface such that $K \equiv -1$, and it is unique up to isometry. A **geodesic** is a curve that is everywhere locally distance minimizing. The hyperbolic plane is also a **unique geodesic metric space**, meaning that between any two points $x, y \in \mathbb{H}^2$, there is a unique geodesic arc from x to y . S is a **hyperbolic surface** if and only if any of the following (equivalent) conditions are true:

- S is a complete, Riemannian surface such that $K \equiv -1$
- $S = G \backslash \mathbb{H}^2$ such that G acts freely, properly discontinuously by isometries on \mathbb{H}^2
- $S = G \backslash \mathbb{H}^2$ such that $G \leq \text{Isom}(\mathbb{H}^2)$ is torsion-free and discrete (i.e. $G \cdot x_0$ discrete for any $x_0 \in \mathbb{H}^2$)

In this subsection, let (S, x) be a hyperbolic surface with basepoint x , and let $p : \mathbb{H}^2 \rightarrow S$ denote the universal covering map. A **closed geodesic** is the unique shortest closed curve in the homotopy class of some closed curve, provided that such a thing exists (i.e. γ is not homotopic to a cusp). A **geodesic loop based at x** is the image under p of a geodesic arc from \tilde{x} to $g \cdot \tilde{x}$ (for some $\tilde{x} \in p^{-1}(x), g \in \pi_1(S, x)$). Note that a geodesic loop α based at x is not necessarily a closed geodesic. However, in certain special cases α may be a closed geodesic, and in this case α will be the unique closed geodesic in the *free* homotopy class of α .

If (γ, x) is a loop based at x , we use the notation $L(\gamma)$ for the length of γ , and $\ell(\gamma) = \ell_x(\gamma) = \ell_{(S, x)}$ for the length of the (unique) shortest geodesic loop based at x that is homotopic to γ leaving the basepoint fixed—this notation is justified by Lemma 5.9 below. In particular, note that $\ell_x(\gamma) \leq L(\gamma)$ with equality holding precisely when γ is a geodesic loop based at x . Let $B_r(x) = \{y \in S : d(x, y) \leq r\}$ denote the closed metric ball of radius r around x ; note that this may not be a topological ball.

Recall from [3] that a closed, connected subset $A \subset S$ is **admissible** if either A consists of exactly one point or else A is a compact connected subset of ∂S . For completeness, we record the result from [3] that we apply below:

Theorem 5.7. *Let S be a hyperbolic surface, let $A, B \subset S$ be admissible subsets, and let $c : [a, b] \rightarrow S$ with $c(a) \in A, c(b) \in B$ be a curve from A to B (A and B need not be different or disjoint). Then the following hold:*

- (1) *In the homotopy class of c with endpoints gliding on A and B there exists a shortest curve γ . This curve is a geodesic arc.*
- (2) *If A and B are points, then γ is unique.*

Theorem 5.7 is Theorem 1.5.3, parts (i) and (vi) from [3]. The following fact is a direct consequence of Theorem 5.7 and says that every element of $\pi_1(S, x)$ may be uniquely represented by a geodesic loop based at x .

Lemma 5.8. *Let $g \in \pi_1(S, x)$ be nontrivial. Then up to based homotopy, there is a unique shortest loop α based at x such that $[\alpha] = g$. Furthermore, α is a geodesic loop.*

Proof. Every such $g \in \pi_1(S, x)$ may be represented by some loop $c : [a, b] \rightarrow S$ with $c(a) = c(b) = x \in S$. Let $A = B = \{x\}$, so A and B are admissible. Applying Theorem 5.7 in this situation yields a unique shortest loop α based at x , and this loop is a geodesic loop. \square

Thus, each element of $\pi_1(S, x)$ has a unique shortest representative $(\alpha, x) \subset (S, x)$ and α is a geodesic loop. This allows us to identify group elements with certain curves (not homotopy classes of curves) on S . We save this in the following lemma for future reference.

Lemma 5.9. *Every element $g \in \pi_1(S, x)$ has a unique shortest representative $(\alpha_g, x) \subset (S, x)$, and α_g is a geodesic loop based at x . In particular, we may abuse notation and write $\alpha_g = [\alpha_g] = g \in \pi_1(S, x)$.*

Recall that there are only finitely many closed geodesics of length less than N for any given $N < \infty$; see, for example, 1.6.11 in [3]. There is an analogous statement for geodesic loops based at $x \in S$, and we provide the elementary proof here:

Proposition 5.10. *Let $N < \infty$ be given. Then the set*

$$\{\alpha \in \pi_1(S, x) : \ell(\alpha) < N\}$$

is finite.

Proof. Although one could mimic the proof of 1.6.11 in [3], we provide our own proof. Fix a preferred lift $\tilde{x} \in \mathbb{H}^2$ of $x \in S$. Let $\Gamma \leq PSL_2(\mathbb{R})$ be a representation of G in $\text{Isom}(\mathbb{H}^2)$, so Γ acts discontinuously on \mathbb{H}^2 . In particular, every closed metric ball $B_N(\tilde{x})$ of radius N centered at \tilde{x} contains only finitely many lifts of the basepoint $x \in S$. A geodesic loop based at x lifts to a geodesic segment between \tilde{x} and another lift of x . But such a geodesic loop has length $\leq N$ if and only if such a lift is contained in $B_N(\tilde{x}) \subset \mathbb{H}^2$, which contains only finitely many lifts of x . Thus, there are only finitely many geodesic loops of length $\leq N$. \square

To prove that subgroups are RS -separable in closed surface groups, we will need the following fact about collections of geodesic loops in S .

Lemma 5.11. *Let (S, x) be an oriented hyperbolic surface with basepoint x . Let A_S denote the set of geodesic loops based at x that are not products of shorter loops based at x . Then:*

- (1) *For every $\alpha \in A_S$, α is simple.*
- (2) *For every $\alpha, \beta \in A_S$, $\alpha \cap \beta = \{x\}$.*
- (3) *$\langle A_S \rangle = \pi_1(S, x)$.*

In particular, if $\pi_1(S, x)$ is finitely generated, then A_S is finite.

Remarks:

- (1) This result is comparable to Lemma 4.4 in Section 4 (Theorem 4.3 in [6]), and in particular, condition (3) essentially follows from Proposition 4.5.
- (2) On a hyperbolic surface, the intersection (away from the basepoint x) of two geodesic loops must be transverse because, upon lifting a neighborhood of the intersection to the universal cover $\mathbb{H}^2 \rightarrow S$, we see that if two geodesics have parallel tangent vectors at their intersection point then they must coincide.

Proof of Lemma 5.11. First we prove (1). Suppose α is not simple and let $c \in \alpha$ be a self-intersection point adjacent on α to x . We will write α as the composition of two shorter loops based at x , which will lead to a contradiction. Let $\eta \subset \alpha$ be one of the two arcs connecting x to c in α , and let β be the other one. Without loss of generality, assume $L(\eta) \leq L(\beta)$. Denote the subloop of α based at c by δ ; see Figure 3.

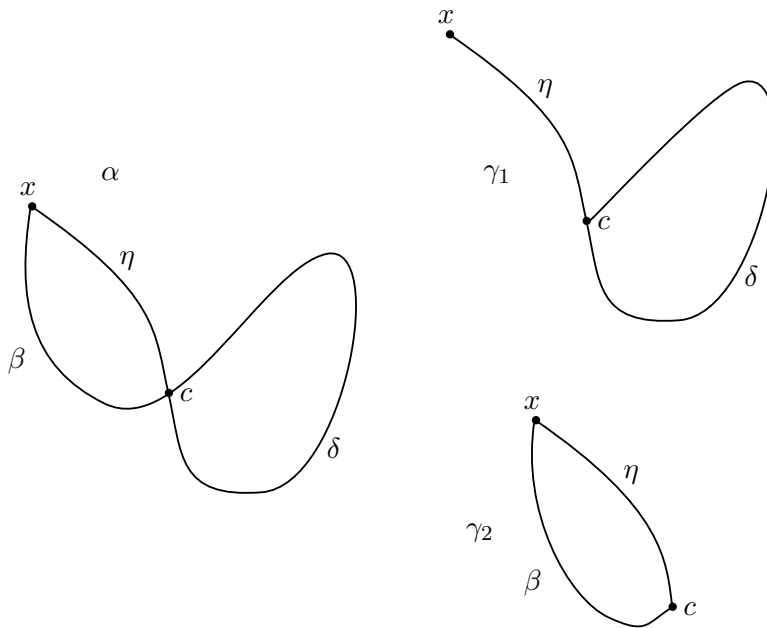


FIGURE 3. Decomposing α into a product of shorter closed loops based at x .

Now write

$$\begin{aligned}
 \alpha &= \eta \circ \delta \circ \beta \\
 &= \eta \circ \delta \circ (\eta^{-1} \circ \eta) \circ \beta \\
 &= \gamma_1 \circ \gamma_2
 \end{aligned}$$

where $\gamma_1 = \eta \circ \delta \circ \eta^{-1}$ and $\gamma_2 = \eta \circ \beta$ are loops based at x . Note that $L(\gamma_1) = 2L(\eta) + L(\delta) \leq L(\alpha)$, although γ_1 is not a geodesic loop based at x since there is a corner at c . Replacing γ_1 with its unique shortest geodesic loop representative, we find that $\ell(\gamma_1) < \ell(\alpha)$. Similarly, $\ell(\gamma_2) < L(\gamma_2) = L(\eta) + L(\beta) < L(\alpha)$. This implies $\alpha \notin A_S$, a contradiction. Therefore α is simple, which proves (1).

Now we prove (2). Assume that $L(\alpha) \leq L(\beta)$. Suppose that α and β intersect somewhere other than x , say at a point $c \in S$, and write $\alpha = \alpha_1 \circ \alpha_2$ and $\beta = \beta_1 \circ \beta_2$ where each arc α_i and β_i has endpoints x and c . Assume further that $L(\alpha_1) \leq L(\alpha_2)$ and $L(\beta_1) \leq L(\beta_2)$. Note that since α and β are geodesic loops, the intersection at c must be transverse. There are two cases:

Case 1: $L(\alpha_1) < L(\beta_1)$. Then $\ell(\beta_1 \alpha_1^{-1}) < L(\beta_1 \alpha_1^{-1}) < L(\beta) = \ell(\beta)$ and $L(\alpha_1 \beta_2) < L(\beta)$. But then $\beta = (\beta_1 \alpha_1^{-1})(\alpha_1 \beta_2)$ is a product of shorter loops, a contradiction.

Case 2: $L(\beta_1) \leq L(\alpha_1)$. Then $\ell(\beta_1 \alpha_2) < L(\beta_1 \alpha_2) \leq \ell(\alpha)$ and $\ell(\alpha_1 \beta_1^{-1}) < \ell(\alpha)$. So we find that $\alpha = (\alpha_1 \beta_1^{-1})(\beta_1 \alpha_2)$ is a product of shorter loops, a contradiction.

Finally, we prove (3). If (3) does not hold, let $\beta \in \pi_1(S, x) \setminus \langle A_S \rangle$ be a shortest geodesic loop. But then, since the only curves on S that are shorter than β represent elements of $\langle A_S \rangle$, we find $\beta \in A_S$, a contradiction. If $\pi_1(S, x)$ is finitely generated, say by $X = \{x_1, \dots, x_n\}$, then each $x_i = w_i([\alpha_{i,j}])$ for some $\alpha_{i,j} \in A_S$, and the collection $\{\alpha_{i,j}\}$ is finite. Let $L = \max_{i,j} L(\alpha_{i,j})$. If $\alpha \in A_S$ with $L(\alpha) > L$, then since $\{[\alpha_{i,j}]\}$ generate, α is a product in $\alpha_{i,j}$, which implies $\alpha \notin A_S$. Therefore A_S is finite by Proposition 5.10. \square

If $(K, x) \subset (S, x)$ is an essential subsurface, then let A_K denote the geodesic loops $(\alpha, x) \subset (K, x)$ which are not products of shorter loops in (K, x) . In certain important cases, $A_K \subset A_S$, and sometimes we may even have $A_K = A_S \cap K$, but this is not true in general; for example, if K is an annular neighborhood of a simple closed loop α not in A_S , then $\alpha \in A_K$ although $\alpha \notin A_S$.

5.4. RS separating subgroups of surface groups. Here we prove that orientable surface groups are LVRSS (Theorem 5.13). The proof is in two main steps: first we use Scott's Theorem to show that, given any finitely generated subgroup H of a surface group, there is a finite-sheeted cover that contains a "large" essential subsurface whose fundamental group is H , and then we show that these large essential subsurfaces are RS separable in that cover.

Theorem 5.12. *Let (Σ, x) be a closed, orientable, hyperbolic surface, and let $H \leq \pi_1(\Sigma, x)$ be a finitely generated infinite-index subgroup. Then there exists a finite-sheeted cover $(\Sigma', x') \rightarrow (\Sigma, x)$ such that $H \leq_{RS} \pi_1(\Sigma', x')$.*

Proof. Consider the cover $(\Sigma_H, x_H) \rightarrow (\Sigma, x)$ corresponding to $H \leq \pi_1(\Sigma, x)$. Note that by Lemma 5.11, the collection A_{Σ_H} generates H and is finite. Let L be bigger than the length of any element of A_{Σ_H} , so that $K_H = B_L(x_H)$ contains a compact core of (Σ_H, x_H) and $\pi_1(K_H, x_H) = H$. Then Proposition 5.6 implies that there exists a finite-sheeted cover $(\Sigma', x') \rightarrow (\Sigma, x)$ where $(K, x_H) \hookrightarrow (K', x') \subset (\Sigma', x')$ embeds isometrically as an essential subsurface of Σ' .

For each $b \in \pi_1(\Sigma', x') \setminus \pi_1(K', x')$, apply Lemma 5.8 to fix its unique geodesic loop representative β , and call this collection of geodesic loops \mathcal{B} . Let $(\Sigma_0, x_0) := (\Sigma', x')$ and $K_0 = K'$. Our goal is to find an RS tower

$$\cdots \rightarrow (\Sigma_n, x_n) \rightarrow (\Sigma_{n-1}, x_{n-1}) \rightarrow \cdots \rightarrow (\Sigma_1, x_1) \rightarrow (\Sigma_0, x_0)$$

such that $\cap \pi_1(\Sigma_n, x_n) = \pi_1(K_0, x_0) = H$.

Since there are only finitely many geodesic loops based at x with length less than any given finite number (Proposition 5.10), there exists a shortest element $\beta_0 \in \mathcal{B}$. First, we claim that β_0 is simple and intersects each $\alpha \in A_{K_0}$ minimally. Since $\beta_0 \notin \pi_1(K_0, x_0)$, we know that $\beta_0 \in A_{\Sigma_0}$. Also, $\beta_0 \notin A_{K_0}$ because every $\alpha \in A_{K_0}$ has $\alpha \in \pi_1(K_0, x_0)$. The hypopaper implies that $K_0 = B_L(x_0)$ implies $A_{K_0} = A_{\Sigma_0} \cap K_0$, and thus $A_{K_0} \sqcup \{\beta_0\} \subset A_{\Sigma_0}$. Therefore Lemma 5.11 implies that β_0 is simple and $\beta_0 \cap \alpha = \{x\}$ for all $\alpha \in A_{K_0}$.

Now β_0 must exit K_0 , and thus $i(\beta_0 \cap \partial K_0) = 2n$ for some $n > 0$. If $n > 1$, then, since geodesic arcs from x_0 to $\partial K_0 = \partial B_L(x_0)$ have length L , and general paths (including intersections of K_0 with geodesic loops that exit K_0) from x_0 to ∂K_0 have length at least L , we could write β_0 as the product of loops shorter than β_0 (some exiting K_0 and others remaining inside K_0), a contradiction. Thus, $|\beta_0 \cap \partial K_0| = 2$.

Now we work with β_0 . If β_0 intersects two distinct components of ∂K_0 , then apply Lemma 5.1 to find a double-cover $\Sigma_1 \rightarrow \Sigma_0$ to which K_0 lifts but β_0 does not. If β_0 intersects a single component of ∂K_0 twice, then write $\beta_0 = \alpha_0 \gamma_0$ for some simple $\alpha_0 \in \pi_1(K_0, x_0)$ and a simple closed curve (γ_0, x_0) freely isotopic to $\gamma \subset S_0$, a connected component of $\Sigma_0 \setminus K_0$. We apply Lemma 5.3 to find a sequence of at most three double covers $\Sigma_1 \rightarrow \Sigma_0'' \rightarrow \Sigma_0' \rightarrow \Sigma_0$ such that $\beta_0 = \alpha_0 \gamma_0$ does not lift to Σ_1 although K_0 does.

Note that lengths of geodesic loops are preserved under taking covers, i.e. if $(\gamma, x_1) \subset (\Sigma_1, x_1)$ is a geodesic loop, and $p_1 : (\Sigma_1, x_1) \rightarrow (\Sigma_0, x_0)$ is the cover, then $\ell_{(\Sigma_1, x_1)}(\gamma) = \ell_{(\Sigma_0, x_0)}(p_1(\gamma))$. Similarly, in any cover (Σ_n, x_n) to which K_0 lifts, it lifts to $K_n = B_L(x_n)$ in the cover. Now for $n \geq 1$, repeat the construction above with the shortest $\beta_n \in \mathcal{B} \cap \Sigma_n$ to get a tower $\cdots \rightarrow (\Sigma_{n+1}, x_{n+1}) \rightarrow (\Sigma_n, x_n) \rightarrow \cdots \rightarrow (\Sigma_0, x_0)$. By Proposition 5.10, for every $\beta \in \mathcal{B}$ there is a cover in the tower to which β does not lift. Thus, we have $\cap \pi_1(\Sigma_n, x_n) = \pi_1(K, x)$ as needed. Furthermore, this is an RS tower because $\pi_1(\Sigma_n, x_n)/\pi_1(\Sigma_{n+1}, x_{n+1})$ is abelian for every $n \geq 0$. \square

Theorem 5.13. *Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g \geq 0$ and $n \geq 0$ boundary components. Then $G = \pi_1(\Sigma)$ is LVRSS.*

Proof. Let $H \leq G$ be finitely generated, where G is the fundamental group of an orientable surface Σ . By Corollary 4.9, we may assume that Σ is closed, that is, $n = 0$. Since finitely generated torsion-free abelian groups are LVRSS, we may further assume that $g \geq 2$.

Thus, assume Σ is a closed, hyperbolic surface, set $G := \pi_1(\Sigma, x)$, and let $H \leq G$ be a finitely generated subgroup. Since all finite-index subgroups are virtually RS-separable, we may assume that H is infinite-index in G , and thus Theorem 5.12 applies, which finishes the proof. \square

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