Summary from Last week:

The complete two-way model:

$$Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}$$

To test for interaction:

$$H_0^{AB}: \text{There is no interaction}$$

This can be restated as:

$$H_0^{AB}: [(\alpha\beta)_{ij} - (\alpha\beta)_{iq}] - [(\alpha\beta)_{sj} - (\alpha\beta)_{sq}] = 0$$
$$\text{for all } i \neq s,\ j \neq q$$

or as

$$H_0^{AB}: [(\alpha\beta)_{i+1,j+1} - (\alpha\beta)_{i+1,j}] - [(\alpha\beta)_{i,j+1} - (\alpha\beta)_{ij}] = 0$$
$$\text{for all } i = 1, \ldots , a-1,\ j = 1, \ldots , b-1$$

If H_0^{AB} is true, then the full model reduces to:

$$Y_{ijt} = \mu^* + \alpha_i^* + \beta_j^* + \varepsilon_{ijt}$$

where
$$\mu^* = [\mu - (\overline{\alpha\beta})_{..}]$$
$$\alpha_i^* = [\alpha_i + (\overline{\alpha\beta})_{i.}]$$
$$\beta_j^* = [\beta_j + (\overline{\alpha\beta})_{.j}]$$

This has the form of a main effects model, but the parameters are different from those in the complete model.

We can test H_0^{AB} using the F-statistic msAB/msE, where

msAB = ssAB/(a-1)(b-1), ssAB = ss E_0^{AB} - ssE.

We obtained a formula for ssAB. This statistic has (a-1)(b-1) and n - ab degrees of freedom.

Examples:

1. The battery experiment

2. The reaction time experiment (pp. 98, 148, 157 of textbook). The data are from a pilot experiment to compare the effects of auditory and visual cues on speed of response. The subject was presented with a "stimulus" by computer, and their reaction time to press a key was recorded. The subject was given either an auditory or a visual cue before the

stimulus. The experimenters were interested in the effects on the subjects' reaction time of the auditory and visual cues and also in the effect of different times between cue and stimulus. The factor "cue stimulus" had two levels, "auditory" and "visual" (coded as 1 and 2, respectively). The factor "cue time" (time between cue and stimulus) had three levels: 5, 10, and 15 seconds (coded as 1, 2, and 3, respectively). The response (reaction time) was measured in seconds.

What next?
This depends on whether or not interaction is significant *and* on what the original questions were in designing the experiment *and* on whether or not the analyzer wishes to engage in data-snooping *and* on the context of the experiment. We will spend a while discussing this.
First, *if* the interaction is deemed not significant, then it is usually desirable to analyze the main effects.

Note: The next section is a replacement of the section "Testing main effects with the complete model" from the notes posted for Friday, March 4.

Testing the contribution of each factor in the complete model (equal sample sizes)

Note: We are still assuming equal sample sizes.

We wish to test whether or not the factor A is needed in the model. Since A is included in two ways, via the α_i's and also via the interaction terms $(\alpha\beta)_{ij}$, we can frame this question as a hypothesis test with null hypothesis

H_0^{A+AB}: Every α_i and every $(\alpha\beta)_{ij} = 0$

and alternate hypothesis

H_a^{A+AB}: At least one of the α_i's or $(\alpha\beta)_{ij}$'s is not zero.

We will again use an F test comparing the full model with the reduced model where all H_0^{A+AB} is true. If sample sizes are equal, it can be shown that the least squares estimate of $E[Y_{ijt}]$ under this new reduced model (i.e, under H_0^{A+AB}) is

$$\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot\cdot\cdot},$$

giving sum of squares for the reduced model

$$ssE_0^{A+AB} = \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{ij\cdot} + \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2,$$

which by appropriate algebraic manipulations becomes

$$ssE_0^{A+AB} = \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{ij\cdot})^2 - br\sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2$$

$$= ssE - br\sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2,$$

so the *sum of squares for treatment factor A* is

$$ssA = ssE_0^{A+AB} - ssE$$

$$= br\sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2$$

$$= (1/br)\sum_{i=1}^{a} (y_{i\cdot\cdot})^2 - (y_{\cdots})^2/abr,$$

which resembles the formula for ssT used to test equality of effects in one-way analysis of variance.

If SSA is the random variable corresponding to ssA, it can be shown that when H_0^{A+AB} is true and sample sizes are equal,

i) $SSA/\sigma^2 \sim \chi^2(a-1)$

ii) SSA and SSE are independent.

Thus, when sample sizes are equal and H_0^{A+AB} is true,

$$\frac{SSA/(a-1)\sigma^2}{SSE/(n-ab)\sigma^2} = \frac{MSA}{MSE} \sim F(a-1,n-ab)$$

So we can use msA/msE as a test statistic, rejecting for large values.

Note: Recall that if we assume that there is no interaction -- that is, that H_0^{AB} is true, then the complete model can be stated as

$$Y_{ijt} = \mu^* + \alpha_i^* + \beta_j^* + \varepsilon_{ijt}$$

where

$$\mu^* = \mu - (\overline{\alpha\beta})_{\cdot\cdot}$$
$$\alpha_i^* = \alpha_i + (\overline{\alpha\beta})_{i\cdot}$$
$$\beta_j^* = \beta_j + (\overline{\alpha\beta})_{\cdot j}$$

The hypothesis, "Factor A has no effect on the mean response," can then be stated as

H_0^A: $\alpha_1^* = \alpha_2^* = \ldots = \alpha_a^*$

If we form an F test for this hypothesis under this model (remembering that we are assuming that there is no interaction), I'm pretty sure we get the same formulas as above, but haven't gone through the details myself.

Similarly, we can form the *sum of squares for treatment factor B* and obtain an F-test based on

$$\frac{SSB/(b-1)\sigma^2}{SSE/(n-ab)\sigma^2} = \frac{MSB}{MSE} \sim F(b\text{-}1, n\text{-}ab)$$

for

H_0^{B+AB}: Every β_j and every $(\alpha\beta)_{ij} = 0$

against the alternate hypothesis

H_a^{B+AB}: At least one of the β_j's or $(\alpha\beta)_{ij}$'s is not zero.

Analysis of Variance Table

For each of the three tests (for interaction, effect of A and effect of B), we have a corresponding sum of squares, ssAB, ssA, and ssB. We also have the error sum of squares, ssE. If add up the formulas for these three sums of squares and do appropriate algebraic manipulations, we will get

$$\text{ssA + aaB + ssAB + ssE} = \sum_i \sum_j \sum_t (y_{ijt} - \bar{y}_{...})^2.$$

This last sum of squares is called the *total sum of squares*, denoted ssT. It can be seen as a measure of the total variability of the data without taking into account either A or B. Similarly, ssE is a measure of the variability taking into account A, B and their interaction; ssA is a measure of the variability taking B into account but not A, and ssB is a measure of the variability taking A into account but not B.

The sums of squares and the additional information used in the tests for A, B and AB are traditionally summarized in an *Analysis of Variance Table* with one line each for A, B, AB, error, and "total sum of squares"

sstot = ssA + ssB + ssAB + ssE

Interpreting ANOVA tests

Interpretation requires thought -- we need to taking into account the purpose of the study, the context, multiple comparisons, and whether or not we are willing to do data snooping. Interpretation can sometimes be frustrating -- for example, what if the test for interaction is significant, but the test for one of the factors is not?

Examples: Battery and reaction time.

Note When sample sizes are unequal, the formulae for the sums of squares are more complicated, and the corresponding random variables are not independent. More on this later.