MODEL AND ANALYSIS FOR RANDOMIZED COMPLETE BLOCK DESIGNS

The *randomized complete block design* (RCBD)

v treatments (They could be treatment combinations.)

b blocks of v units, chosen so that units within a block are alike (or at least similar) and units in different blocks are substantially different. (Thus the total number of experimental units is n = bv.)

The v experimental units within each block are randomly assigned to the v treatments. (So each treatment is assigned one unit per block.)

Model:

$Y_{hi} = \mu + \theta_h + \tau_i + \varepsilon_{hi}$

$\varepsilon_{hi} \sim N(o,\sigma^2)$

ε_{hi}'s independent

where

Y_{hi} is the random variable representing the response for treatment i observed in block h,

μ is a constant (which may be thought of as the overall mean – see below)

θ_h is the (additive) effect of the h^{th} block (h = 1, 2, … , b)

τ_i is the (additive) effect of the i^{th} treatment (i = 1, 2, … , v)

ε_{hi} is the random error for the i^{th} treatment in the h^{th} block.

(Why is there no subscript t for observation number?)

Note:

- This model formally looks just like a two-way main effects model – but you need to remember that there is just one factor plus one block; the randomization is just within each block. So we don't have the conditions for a two-way analysis of variance.
- Like the main-effects model, this is an additive model that does not provide for any interaction between block and treatment level – it assumes that treatments have the same effect in every block, and the only effect of the block is to shift the mean response up or down. If interaction between block and factor is suspected, then either a transformation is needed to remove interaction before using this model, or a design with more than one observation per block-treatment combination must be used. (Trying to add an interaction term in the RCBD would create the same problem as is encountered in two-way ANOVA with one observation per cell: the degrees of freedom for the error is zero, so the method of analysis breaks down.)
- This is an over-specified model; the additional constraints $\sum_{h=1}^{b}\theta_h = 0$ and $\sum_{i=1}^{v}\tau_i = 0$, are typically added, so that the treatment and block effects are thought of as deviations from the overall mean.
- There is an alternate *means model* $Y_{hi} = \mu_{ih} + \varepsilon_{hi}$, where $\mu_{ih} = \mu + \theta_h + \tau_i$.

- Note that the i[th] treatment mean is $\mu_i = \frac{1}{b}\sum_{h=1}^{b}(\mu + \theta_h + \tau_i)$. Assuming the constraint $\sum_{h=1}^{b}\theta_h = 0$, this gives $\mu_i = \mu + \tau_i$

Estimating and Analysis:

Least squares fits: Since the model is formally the same as the main-effects model, the process of finding least squares is the same as for the latter model, yielding estimates (with notation appropriately changed)

$\mu\hat{} = \bar{y}_{\bullet\bullet}$
$\theta_h\hat{} = \bar{y}_{h\bullet} - \bar{y}_{\bullet\bullet}$
$\tau_i\hat{} = \bar{y}_{\bullet i} - \bar{y}_{\bullet\bullet}$
$y_{hi}\hat{} = \mu\hat{} + \theta_h\hat{} + \tau\hat{} =$
$= \bar{y}_{h\bullet} + \bar{y}_{\bullet i} - \bar{y}_{\bullet\bullet}$

Thus the error sum of squares for this model is

ssE =

As with the two-way main effects model, msE = ssE/(b-1)(v-1) is an unbiased estimator of σ^2. (Note that (b-1)(v-1) = bv – b – v + 1 = n – b – v + 1, since n = bv.)

Model checking: Look especially for potential problems with the normality assumption, unequal error variance by block or treatment, and treatment-block interaction, which is suggested by a curvilinear pattern in the plot of residuals vs. fits. (Note that since there is one observation per block, treatment level combination, there is no way to check the equal variance assumption at that fine a level.)

Hypothesis test and Analysis of Variance Table: We are interested in testing equality of treatment means. Thus we wish to test the null hypothesis

H_0: $\mu_1 = \mu_2 = \ldots = \mu_v$

against the alternate

H_a: $\mu_i \neq \mu_j$ for at least one pair i,j.

Note: From the last remark above, we can restate the hypotheses as

H_0: $\tau_1 = \tau_2 = \ldots = \tau_v = 0$

and

H_a: at least one $\tau_i \neq 0$.

We can construct an F-test in the usual manner: We consider the submodel corresponding to the null hypothesis, namely

$Y_{hi} = \mu + \theta_h + \varepsilon_{hi}$

This has least squares fits

$(y_{hi}\hat{})_0 = (\bar{y}_{\bullet\bullet}) - (\bar{y}_{h\bullet} - \bar{y}_{\bullet\bullet}) = \bar{y}_{h\bullet}$

and hence error sum of squares

$ssE_0 =$

The difference $ssT = ssE_0 - ssE$ is called the *sum of squares for treatment*. Our test statistic for H_0 is

$$\frac{ssT/(v-1)}{ssE/(b-1)(v-1)}.$$

As usual, the numerator is denoted msT (with v-1 degrees of freedom) and the denominator msE (with (b-1)(v-1) degrees of freedom, as mentioned above. The test statistic has an F distribution with v-1 degrees of freedom in the numerator, (b-1)(v-1) in the denominator.

Note:

- The above test is the same as the F-test for the treatment factor we would get by two-way ANOVA considering treatment and block as two factors in a main effects model. Thus we can test our hypothesis by using a two-way ANOVA main-effects software routine.
- We can define ssB and msB (using b-1 degrees of freedom), but we *don't* get a legitimate F-test for the null hypothesis "No block effect," since the conditions for proving that the would-be test statistic has an F-distribution are not met, because the blocks are chosen, not randomly assigned.
- Nonetheless, the ratio msB/msE can be considered as an informal measure of the effect of the blocking factor – if the ratio is large, that suggests that the blocking "factor" has a large effect, and that the variance reduction obtained by blocking was probably helpful in by improving the precision in the comparison of treatment means.
- The algebra works out to show that ssTot = ssB + ssT + ssE, and the degrees of freedom add accordingly.

Contrasts: In the RCBD, all contrasts (with coefficient sum zero) in the treatment effects τ_i are estimable, and the techniques of Chapter 4 still apply, with the following observed:

- The estimate of τ_i is $\tau_i\hat{} = \bar{y}_{\bullet i} - \bar{y}_{\bullet\bullet}$, but since in a contrast $\sum c_i\tau_i$, we have $\sum c_i = 0$, the estimate of the contrast is $\sum c_i \bar{y}_{\bullet i}$
- The number of replicates is equal to the number b of blocks
- The error degrees of freedom are (b-1)(v-1).
- The msE used is the one obtained by the block design analysis.

Example: A hardness testing machine operates by pressing a tip into a metal test "coupon." The hardness of the coupon can be determined from the depth of the resulting depression. Four tip types are being tested to see if they produce significantly different readings. However, the coupons might differ slightly in their hardness (for example, if they are taken from ingots produced in different heats). Thus coupon is a nuisance factor, which can be treated as a blocking factor. Since coupons are large enough to test four tips on, a RCBD can be used, with one coupon as a block. Four blocks were used. Within each block (coupon) the order in which the four tips were tested was randomly determined. The results (readings on a certain hardness scale) are shown in the following table:

	Test Coupon			
Type of Tip	**1**	**2**	**3**	**4**
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

Comment: From the table, the type of design is not apparent – in particular, the table does not show the order in which the observations were made, hence does not show the randomization. However, data are often presented in such a table, for reasons of economy of space or whatever.

We wish to test

H_0: All tips give the same mean reading

against the alternative

H_a : At least two tips give different mean readings.

Our pre-planned analysis will be to test this hypothesis at the .01 level, then if the hypothesis is rejected, to form confidence intervals for pairwise differences at a family rate of 99%, giving an overall confidence/significance level of 98%.

We can run the data on Minitab under Balanced ANOVA in exactly the same way we would run a two-way main effects model. The output is:

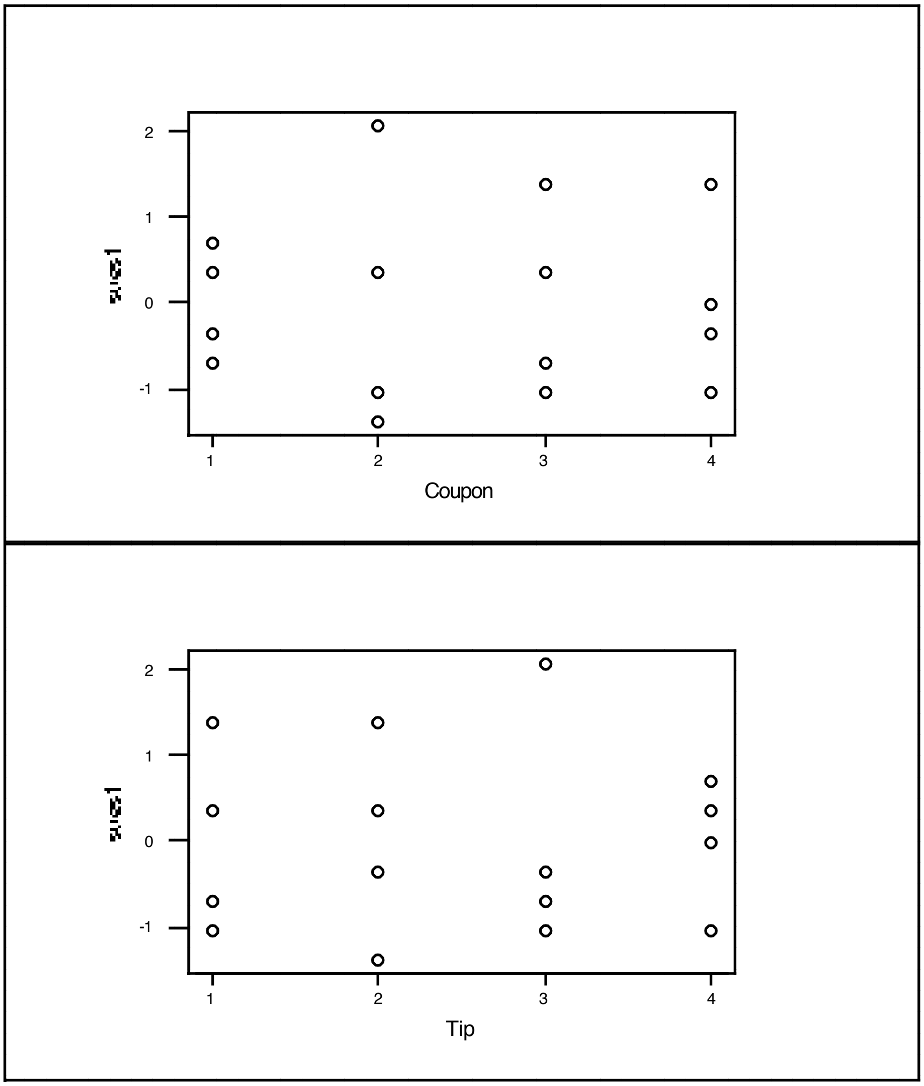
```
Analysis of Variance for hard

Source      DF        SS         MS       F      P
Coupon       3   0.82500    0.27500   30.94  0.000
Tip          3   0.38500    0.12833   14.44  0.001
Error        9   0.08000    0.00889
Total       15   1.29000
```

Note that degrees of freedom and sums of squares behave as expected.

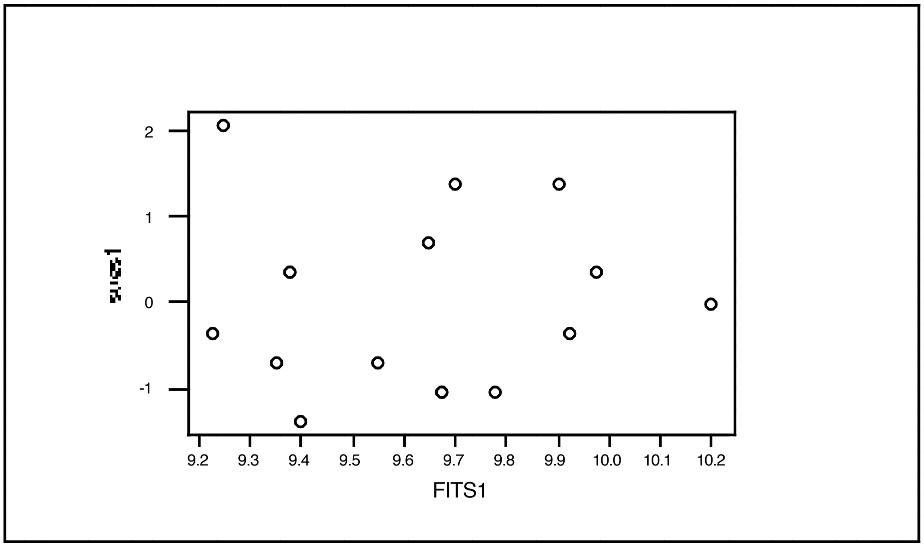
Before testing, we check the model.

The plots of standardized residuals vs blocks and factor levels show one moderate outlier for coupon 2, tip 3. However, the standardized residual value of 2.05 isn't really that unusual with 16 observations.

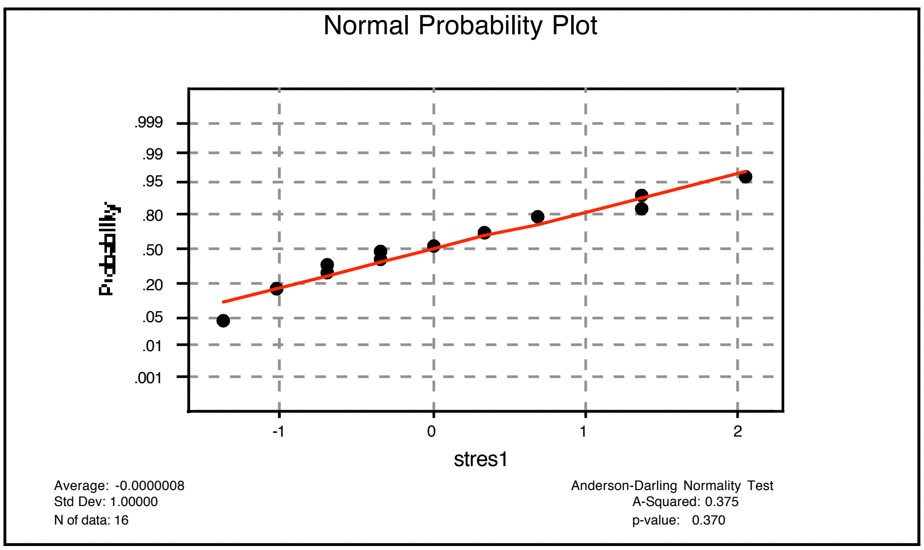


There is also a suggestion of unequal variance by block. Calculating standard deviation by block gives smallest 0.625 (coupon 1) and largest 1.556 (Coupon 2). This seems problematical from the point of view of the rule of thumb for variance ratio, but as shown earlier by simulations, is actually consistent with equal variance for samples this small (4 observations). (Note that large scatter of residuals for a single coupon could indicate that that the coupon is of non-uniform hardness. Here, the possible problem seems to be the possibility of one coupon with smaller variance, which seems less problematical then one with larger variance.)

The plot of standardized residuals vs. fits shows no apparent patterns to suggest either interaction between block and tip or a relationship between mean and variance.



The normal probability plot below seems consistent with normality of errors.



Thus we proceed with inference:

The p-value for our hypothesis test is 0.001, prompting us to reject the null hypothesis of no difference.

We thus proceed to form confidence intervals for differences in effect of tip. (Note that a glance at the data suggests that tip 4 tends to give higher readings; we will see whether or not the confidence intervals suggest that this is more than just chance variability.) Note (see more below) that we cannot use Minitab's option of obtaining the CI's doing one-way analysis of variance – the msE is wrong. The Tukey msd is

$$\text{msd} = w_T\sqrt{msE\left(\frac{1}{4}+\frac{1}{4}\right)} = [q(4, 9, 0.01)/\sqrt{2}\,]\sqrt{0.00889\left(\frac{1}{2}\right)}$$

$$= (5.96/\sqrt{2})(0.0667) = 0.281$$

Using Descriptive Statistics, we calculate the estimates $\bar{y}_{\bullet i}$ to be 9.575, 9.600, 9.450, and 9.875, for i = 1, 2, 3, and 4, respectively. So the centers of the simultaneous 98% confidence intervals for the pairwise difference contrasts are:

Contrast	**Center of CI**
τ_1-τ_2	9.575 – 9.600 = -0.025
τ_1-τ_3	9.575 – 9.450 = 0.125
τ_1-τ_4	9.575 – 9.875 = -0. 300
τ_2-τ_3	9.600 – 9.450 = 0.150
τ_2-τ_4	9.600 – 9.875 = -0.275
τ_3-τ_4	9.450 – 9.875 = - 0.425

Comparing with the msd, we can see that we have the mean for tip 4 significantly different from the means for the other tips, but no significant differences between the means of the other three tips. This is what we suspected from the data.

Note:
1. Of course, in a real experiment, we would investigate Bonferroni and Scheffe methods to see if they might give smaller confidence intervals.
2. In the Analysis of Variance table, we see that the msB (the mean square for the blocks -- coupons) is 0.27500, which is 30.94 times the msE of 0.00889. This suggests that indeed blocking helped get more precise estimates than would be obtained without it. To follow up on this idea, let us suppose that we had obtained the same data from a completely randomized design with the single factor "tip" and 4 replications for each level of tip. Running one-way ANOVA on the data, and also using Tukey's method for the confidence intervals at family error rate 4%, gives output

```
Analysis of Variance on hard
Source     DF        SS        MS        F       p
Tip         3    0.3850    0.1283     1.70   0.220
Error      12    0.9050    0.0754
Total      15    1.2900

Tukey's pairwise comparisons

    Family error rate = 0.0100
Critical value = 5.50
```

```
Intervals for (column level mean) - (row level mean)

             1         2         3

     2   -0.7802
          0.7302

     3   -0.6302   -0.6052
          0.8802    0.9052

     4   -1.0552   -1.0302   -1.1802
          0.4552    0.4802    0.3302
```

Notice that we are not detecting any difference at all! The msE is 0.075, several times larger than the msE of 0.00889 obtained by blocking. This translates to a ratio of variance estimates of over 70. This supports the idea that blocking is a variance reduction technique and explains why block designs are so often used.

3. Be sure to read Examples 10.4.1 and 10.4.2, and Section 10.5 for more examples of RCBD's and their analysis.