INFERENCE FOR ONE-WAY ANOVA

To test equality of means for different treatments/levels, we can use the null hypothesis

H_0: $\mu_1 = \mu_2 = \ldots = \mu_v$

Rephrase:

1. In terms of effects: ________________________________

2. In terms of differences of effects: __________________________________

3. In terms of contrasts $\tau_i - \bar{\tau}$, where $\bar{\tau} = \frac{1}{v}\sum_{i=1}^{v}\tau_i$: ________________________

The *treatment degrees of freedom* is the minimum number of equations needed to state the null hypothesis, in other words _______________.

Alternate hypothesis: H_a: _____________________________

Idea of the test: Compare ssE under the *full* model (with all parameters) with the error sum of squares ssE_0 under the *reduced* model -- i.e., the one assuming H_0 is true.

To calculate ssE_0: If H_0 is true, let τ be the common value of the τ_i's. Then the reduced model is

- $Y_{it} = \mu + \tau + \varepsilon_{it}^0$
- $\varepsilon_{it}^0 \sim N(0, \sigma^2)$
- the ε_{it}^0's are independent,

where ε_{it}^0 denotes the itth error in the reduced model.

To find ssE_0, we use least squares to minimize g(m) = $\sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - m)^2$:

$$g'(m) = \sum_{i=1}^{v}\sum_{t=1}^{r_i} 2(-1)(y_{it} - m) = 0,$$

which yields estimate $\bar{y}_{..}$ for $\mu + \tau$ -- that is, the least squares estimate of $\mu + \tau$ is $(\mu + \tau)^\wedge = \bar{y}_{..}$. (By abuse of notation, we call this $\hat{\mu} + \hat{\tau}$). So

$$ssE_0 = \sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - \bar{y}_{..})^2,$$

which can be shown (proof might be homework) to equal $\sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}_{..})^2$

Note that ssE and ssE_0 can be considered as minimizing the same expression, but over different sets: ssE minimizes $\sum_{i=1}^{t}\sum_{t=1}^{r_i}(y_{it} - m - t_i)^2$ over the set of all v + 1-tuples (m, t_1, t_2, … , t_v), whereas ssE_0 can be considered as minimizing the same expression over the subset where all t_i's are zero. Thus ssE_0 must be at least as large as ssE: $ssE_0 \geq ssE$.

However, if H_0 is true, then ssE and ssE_0 should be about the same. This suggests the idea of using the ratio (ssE_0-ssE)/ssE as a test for the null hypothesis: If H_0 is true, this ratio should be small; so an ususually large ratio would be reason to reject the null hypothesis.

The difference ssE_0-ssE is called the *sum of squares for treatment,* or *treatment sum of squares*, denoted ssT. Using the alternate expressions for ssE_0 and ssE, we have:

$$ssT = ssE_0 - ssE = \sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - n(\bar{y}_{..})^2 - \left(\sum_{i=1}^{v}\sum_{t=1}^{r_i} y_{it}^2 - \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2\right)$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet})^2 - n(\bar{y}_{..})^2$$

$$= \sum_{i=1}^{v} \frac{(y_{i\bullet})^2}{r_i} - \frac{(y_{..})^2}{n} \qquad \text{(using definitions)}$$

$$= \sum_{i=1}^{v} r_i(\bar{y}_{i\bullet} - \bar{y}_{..})^2 \qquad \text{(possible homework)}$$

This last expression can be considered as a "between treatments" sum of squares --- we are comparing each treatment sample mean $\bar{y}_{i\bullet}$ with the grand (overall) mean $\bar{y}_{..}$ By contrast, our denominator, ssE = $\sum_{i=1}^{v}\sum_{t=1}^{r_i}(y_{it} - \bar{y}_i)^2$ is a "within treatments" sum of squares: it compares each value with the mean for the treatment group from which the value was obtained.

Using the model assumptions, it can be proved that:

- $ssE/\sigma^2 \sim \chi^2(n - v)$
- If H_0 is true, $ssT/\sigma^2 \sim \chi^2(v - 1)$
- If H_0 is true, then ssT and ssE are independent.

Thus, <u>if</u> H_0 is true,

$$\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)} \sim F_{v-1,n-v}.$$

Now $\frac{ssT/\sigma^2(v-1)}{ssE/\sigma^2(n-v)}$ simplifies to $\frac{ssT/(v-1)}{ssE/(n-v)}$, which we can calculate from our sample.

We originally wanted to test ssT/ssE, but $\frac{ssT/(v-1)}{ssE/(n-v)}$ is just a constant multiple of ssT/ssE, so is good enough for our purposes: $\frac{ssT/(v-1)}{ssE/(n-v)}$ will be unusually large exactly when ssT/ssE is unusually large. Thus, we can use an F test, with test statistic $\frac{ssT/(v-1)}{ssE/(n-v)}$, to test our hypothesis.

Note: We can look at ssT/(v-1) and ssE/(n-v) as we did in the equal-variance, two-sample t-test: ssE/(n-v) is a pooled estimate of the common variance σ^2, and if H_0 is true, then ssT/(v - 1) can be regarded as an estimate of σ^2.

Notation: ssT/(v-1) is called msT (*mean square for treatment* or *treatment mean square* and ssE/(n-v) is called msE (*mean square for error* or *error mean square*). So the test statistic is F = msT/msE.