CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS (Section 17.3.5)

These play the role for random effects that confidence intervals for contrasts play for fixed effects.

Confidence intervals for σ^2: These are constructed just as for fixed effects; see Section 3.4.6 or the class notes *Choosing Sample Sizes*.

Confidence intervals involving σ_T^2: Three types of confidence intervals are of interest: for σ_T^2, for σ_T^2/σ^2, and for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$. The first cannot be done exactly, so we'll take that last.

Confidence intervals for σ_T^2/σ^2: We use the fact (see notes *Testing for Treatment Effect as a Proportion of Error Variance)* that

$$\frac{MST\big/\left(c\sigma_T^2+\sigma^2\right)}{MSE\big/\left(\sigma^2\right)} \sim \text{F(v-1, n-v)},$$

where c is a certain constant defined in terms of n,v, and the r_i; c = r if the design is balanced) (See notes *Random Effects Models* or Section 17.3)

If we want a (1-α)100% CI for σ_T^2/σ^2, take

f_1 = F(v-1, n-v, 1- α/2) (so that there is area α/2 to the left of f_1 in the F(v-1, n-v) distribution), and

f_2 = F(v-1, n-v, α/2) (so that there is area α/2 to the right of f_2 in the F(v-1, n-v) distribution). [Draw a picture!]

Then

$$\text{Prob}\left(f_1 \le \frac{MST\big/\left(c\sigma_T^2+\sigma^2\right)}{MSE\big/\left(\sigma^2\right)} \le f_2\right) = 1-\alpha,$$

or equivalently,

$$\text{Prob}\left(f_1 \le [MST/MSE]\,[\sigma_T^2/(\sigma_T^2+\sigma^2)] \le f_2\right) = 1-\alpha,$$

The left inequality is equivalent to

$$(c\sigma_T^2+\sigma^2)/\sigma^2 \le (MST/MSE)(1/f_1), \text{ or}$$

$$c(\sigma_T^2/\sigma^2) + 1 \le (MST/MSE)(1/f_1),$$

which is equivalent to

$$c(\sigma_T^2/\sigma^2) \le (MST/MSE)(1/f_1) - 1$$

The right inequality is equivalent to

$$(MST/MSE)(1/f_2) \le (c\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2)/\boldsymbol{\sigma}^2 = c(\boldsymbol{\sigma_T}^2/\boldsymbol{\sigma}^2) + 1,$$

which is equivalent to

$$(MST/MSE)(1/f_2) - 1 \le c(\boldsymbol{\sigma_T}^2/\boldsymbol{\sigma}^2)$$

So

$$\text{Prob}\ ((1/c)[\ (MST/MSE)(1/f_2) - 1] \le \boldsymbol{\sigma_T}^2/\boldsymbol{\sigma}^2 \le (1/c)[\ (MST/MSE)(1/f_2) - 1]) = 1 - \alpha.$$

Thus $((1/c)[\ (msT/msE)(1/f_2) - 1]\ ,\ (1/c)\ (msT/msE)(1/f_2) - 1)$ is the desired confidence interval. (This means ____)

Note: Conceivably the left hand endpoint could be less than 0, which is unrealistic, If it is < 0, do *not* give in to the temptation to replace it by zero; that would give the false impression of a smaller confidence interval than warranted.

Example: Use the loom data to find a 95% confidence interval for $\boldsymbol{\sigma_T}^2/\boldsymbol{\sigma}^2$.

Confidence intervals for $\boldsymbol{\sigma_T}^2/(\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2)$ *= the proportion of the total variance if the response attributable to the treatment level*: Such confidence intervals are readily obtained from confidence intervals for $\boldsymbol{\sigma_T}^2/\boldsymbol{\sigma}^2$ as follows. Divide both numerator and denominator of $\boldsymbol{\sigma_T}^2/(\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2)$ by $\boldsymbol{\sigma}^2$ to obtain

$$\boldsymbol{\sigma_T}^2/(\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2) = \frac{\sigma_T^2/\sigma^2}{\left(\sigma_T^2/\sigma^2\right) + 1} = f(\sigma_T^2/\sigma^2),\ \text{where } f(x) = x/(x + 1)) = \frac{1}{1 + \frac{1}{x}}.$$

From the last formula for f(x), we can see that f(x) is an increasing function of x. Thus if (a,b) is a $(1-\alpha)100\%$ confidence interval for σ_T^2/σ^2, then (f(a), f(b)) = (a/(a + 1), b/(b + 1)) is a $(1-\alpha)100\%$ confidence interval for $\boldsymbol{\sigma_T}^2/(\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2)$.

Example: With the loom data, find a 95% confidence interval for $\boldsymbol{\sigma_T}^2/(\boldsymbol{\sigma_T}^2 + \boldsymbol{\sigma}^2)$.

Confidence intervals for $\boldsymbol{\sigma_T}^2$: There is no exact method. There are several approximate methods. Here is one. It is useful if $\boldsymbol{\sigma_T}^2$ is not too small, and is adaptable to more complicated models.

Recall that $U = (1/c)(MST - MSE)$ is an unbiased estimator of $\boldsymbol{\sigma_T}^2$. If we knew its distribution, we could use that to get confidence intervals for $\boldsymbol{\sigma_T}^2$ in the usual way. However, it does not have a tractable distribution. But it *is* true that

$U/\boldsymbol{\sigma_T}^2 \approx \chi^2(x)/x$, where

$$x \approx \frac{(msT - msE)^2}{(msT)^2/(v-1) + (msE)^2/(n-v)}$$

(Note: This formula is given correctly on p. 605 of the text, but incorrectly on p. 600.)

x is not usually an integer, so we need to interpret degrees of freedom in the χ^2 distribution as the parameter in a formula for the pdf. (This is analogous to the two-sample, unequal variance t-test.)
Thus (Draw a picture!)

$$P(\ \chi^2(x, 1-\alpha/2) \prec xU/\boldsymbol{\sigma_T^2} \prec \chi^2(x, \alpha/2)\ ) \approx 1-\alpha,$$

where $\prec$ means "is less than or approximately equal to", and $\chi^2(x,\beta)$ is the value with proportion β of the $\chi^2(x)$ distribution to its *right*.

The left and right approximate inequalities are, respectively, equivalent to

$$\boldsymbol{\sigma_T^2} \prec xU/\chi^2(x, 1-\alpha/2) \qquad \text{and} \qquad \boldsymbol{\sigma_T^2} \succ xU/\chi^2(x, \alpha/2).$$

Thus if

$u = (1/c)(msT - msE)$ (which is our estimate for $\boldsymbol{\sigma_T^2}$), then

$(xu/\chi^2(x, \alpha/2), xu/\chi^2(x, 1-\alpha/2))$ is an approximate $(1-\alpha)100\%$ confidence interval for $\boldsymbol{\sigma_T^2}$.

Example: With the loom data, find a 95% confidence interval for $\boldsymbol{\sigma_T^2}$.