

## TWO OR MORE RANDOM EFFECTS

**Example:** The factors that influence the breaking strength of a synthetic fiber are being studied. Four production machines and three operators are randomly selected. A two-way factorial experiment is run, with two observations per treatment combination, using raw material from the same production batch, with breaking strength as response.

Here we have two random factors: Interest is in the variability of breaking strength over the range of machines and operators; machines and operators for the experiment are randomly chosen.

**The two-way complete model for two random effects:** There are two random factors, A with a levels and B with b levels. The design is completely randomized, with  $r_{ij}$  observations at treatment combination “level i of A and level j of B.” The complete model is:

$$Y_{ijt} = \mu + A_i + B_j + (AB)_{ij} + \varepsilon_{ijt},$$

where:

$$\text{Each } A_i \sim N(0, \sigma_A^2) \quad \text{Each } B_j \sim N(0, \sigma_B^2)$$

$$\text{Each } (AB)_{ij} \sim N(0, \sigma_{AB}^2) \quad \text{Each } \varepsilon_{ijt} \sim N(0, \sigma^2)$$

The  $A_i$ 's,  $B_j$ 's,  $(AB)_{ij}$ 's, and  $\varepsilon_{ijt}$ 's are all mutually independent random variables.

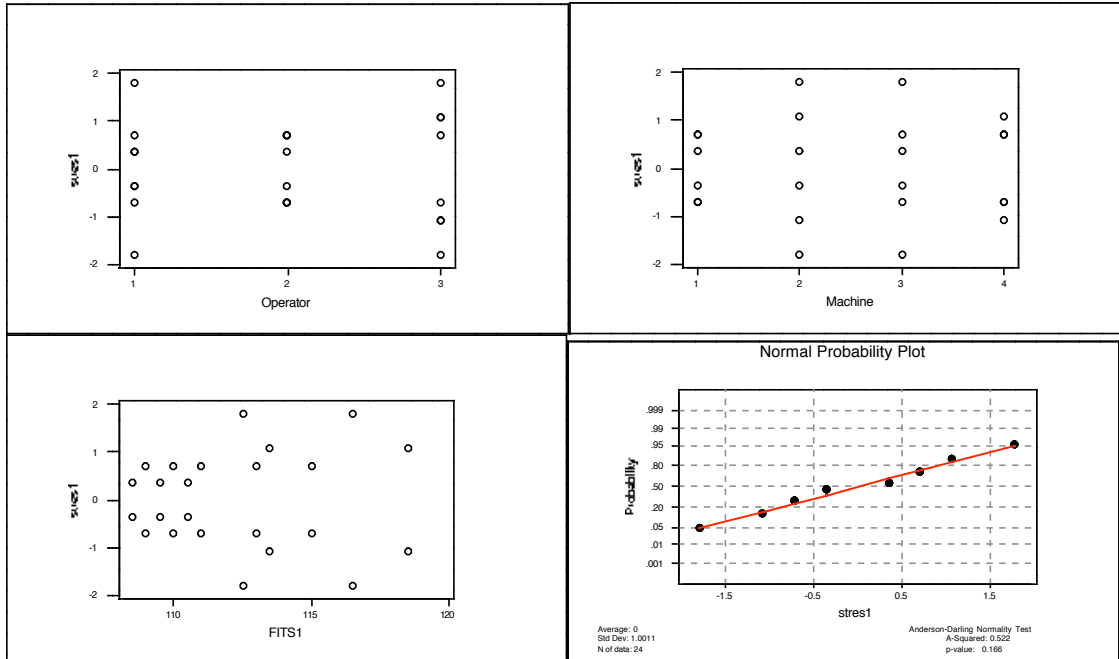
Compare and contrast with both the 1-way random effect and 2-way fixed effect complete models.

Note: 1. If all  $r_{ij}$ 's have the same value  $r$ , then we have a *balanced* design.

2. If we omit the interaction terms  $(AB)_{ij}$ , then we obtain the *two-way main effects model for two random effects*.

Least squares gives exactly the same estimates as for the two-way complete fixed effects model. Thus we can form residuals, sums of squares, and mean squares (in particular,  $msE$ ,  $msAB$ ,  $msA$ ,  $msB$ ) using the same software routines as for the two-way complete fixed effects model. In particular, we can use residuals to check some of the model assumptions. As in the one random effect model, normality of the  $A_i$ 's,  $B_j$ 's, and  $(AB)_{ij}$ 's is important – *but we can't really check this with residuals*, since there are few levels of each treatment factor, and the cell averages are *not* independent.

*Example:* For the fiber strength data, the residual plots and related information that can be used to check the model are:



Max/min standard deviations

By operator 1.55, 2.70

By machine 2.32, 4.46

Calculations using the model assumptions give:

$$E(Y_{ijt}) = \mu \quad \text{Var}(Y_{ijt}) = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma^2 \text{ (using independence!)}$$

What would the null and alternate hypotheses to test for interaction be?

$H_0^{AB}$ :

$H_a^{AB}$ :

We can also consider tests with hypotheses

$$H_0^A: \sigma_A^2 = 0 \quad \text{vs } H_a^A: \sigma_A^2 > 0 \quad \text{and } H_0^B: \sigma_B^2 = 0 \quad \text{vs } H_a^B: \sigma_B^2 > 0,$$

But, for example, “A has no effect” is *not* the same as “ $\sigma_A^2 = 0$ ,” since A might have an effect through interaction. So “A has no effect” means that *both*  $\sigma_A^2$  and  $\sigma_{AB}^2$  are zero.

Recall that for one random effect, the hypothesis test for  $\sigma_T^2$  used the facts that  $E(\text{MSE}) = \sigma^2$  and  $E(\text{MST}) = c\sigma_T^2 + \sigma^2$ . Analogous calculations (details omitted) for the complete two-way random effects model, assuming a balanced design, yield

$$E(\text{MSA}) = br\sigma_A^2 + r\sigma_{AB}^2 + \sigma^2$$

$$E(\text{MSB}) = ar\sigma_B^2 + r\sigma_{AB}^2 + \sigma^2$$

$$E(\text{MSAB}) = r\sigma_{AB}^2 + \sigma^2$$

$$E(\text{MSE}) = \sigma^2.$$

(Note the pattern of the terms. Also note that  $br$  is the number of observations at each level of A, and similarly for other coefficients.)

The test statistics are constructed as follows:

For  $H_0^{AB}$ : If  $H_0^{AB}$  is true, then  $E(\text{MSAB}) = E(\text{MSE})$ . If  $H_0^{AB}$  is false, then  $E(\text{MSAB}) > E(\text{MSE})$ . Using the same reasoning as for one random effect,  $\text{MSAB}/\text{MSE}$  is an appropriate test statistic, and has an  $F((a-1)(b-1), ab(r-1))$  distribution.

For  $H_0^A$ : If  $H_0^A$  is true, then  $E(\text{MSA})$  might *not* equal  $E(\text{MSE})$  – but it *does* equal  $\text{MSAB}$ . So similar reasoning to the above shows that  $\text{MSA}/\text{MSAB}$  is a suitable test statistic for  $H_0^A$ . It has an  $F(a-1, (a-1)(b-1))$  distribution. *Note that the test statistic is not the same as in the fixed effects two-way complete model.*

For  $H_0^B$ : Similar reasoning leads to  $\text{MSB}/\text{MSAB} \sim F(b-1, (a-1)(b-1))$  as test statistic. *Again, the test statistic is not the same as in the fixed effects two-way model.*

*Comments:* 1. The question of whether to test  $H_0^A$  and  $H_0^B$  if we reject  $H_0^{AB}$  arises as with the two-way complete fixed effects model. But here we have an alternative interpretation: We can interpret  $H_0^A$  as saying the variance component  $\sigma_A^2$  is zero, rather than saying A has no effect.

2. As with the two-way complete fixed effects model, if we fail to reject  $H_0^{AB}$ , we should continue our testing within the complete model, rather than switching to the main effects model for the same data.

*Example:* Fiber breaking strength – Run the data as both fixed effect and random effect to see the difference in the test statistics.

I. As fixed effect:

Source	DF	SS	MS	F	P
Operator	2	160.333	80.167	21.14	0.000
Machine	3	12.458	4.153	1.10	0.389
Operator*Machine	6	44.667	7.444	1.96	0.151
Error	12	45.500	3.792		
Total	23	262.958			

II. Designating factors as “random”

Source	DF	SS	MS	F	P
Operator	2	160.333	80.167	10.77	0.010
Machine	3	12.458	4.153	0.56	0.662
Operator*Machine	6	44.667	7.444	1.96	0.151
Error	12	45.500	3.792		
Total	23	262.958			

**Other models with two or more random effects:** The pattern of expected mean squares continues, and leads us to the appropriate tests. Examples:

1) *Two-way main effects with random factors:*  $Y_{ijt} = \mu + A_i + B_j + \varepsilon_{ijt}$ , where

each  $A_i \sim N(0, \sigma_A^2)$ , each  $B_j \sim N(0, \sigma_B^2)$ , each  $\varepsilon_{ijt} \sim N(0, \sigma^2)$ , and the  $A_i$ 's,  $B_j$ 's, and  $\varepsilon_{ijt}$ 's are all mutually independent random variables. Expected mean squares are:

$$E(\text{MSA}) = br\sigma_A^2 + \sigma^2$$

$$E(\text{MSB}) = ar\sigma_B^2 + \sigma^2$$

$$E(\text{MSE}) = \sigma^2.$$

If  $H_0^A: \sigma_A^2 = 0$  is true, then  $E(\text{MSA}) = \underline{\hspace{2cm}}$ , so our test statistic is  $\underline{\hspace{2cm}}$ .

2) *Three way complete random factor model:*

$$Y_{ijkl} = \mu + A_i + B_j + C_k + (AB)_{ij} + (BC)_{jk} + (AC)_{ik} + (ABC)_{ijk} + \varepsilon_{ijkl},$$

with the appropriate normality and independence conditions. Expected mean squares are

$$E(\text{MSA}) = rbc\sigma_A^2 + rc\sigma_{AB}^2 + rb\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma^2$$

$E(\text{MSB})$  and  $E(\text{MSC})$  are similar

$$E(\text{MSAB}) = rc\sigma_{AB}^2 + r\sigma_{ABC}^2 + \sigma^2$$

$E(\text{MSBC})$  and  $E(\text{MSAC})$  are similar

$$E(\text{MSABC}) = r\sigma_{ABC}^2 + \sigma^2$$

$$E(\text{MSE}) = \sigma^2.$$

i) If  $H_0^{ABC}: \sigma_{ABC}^2 = 0$  is true, then  $E(\text{MSABC}) = \underline{\hspace{2cm}}$ , so our test statistic is  $\underline{\hspace{2cm}}$ . (It has an F-distribution with appropriate degrees of freedom.)

ii) If  $H_0^{AB}: \sigma_{AB}^2 = 0$  is true, then  $E(\text{MSAB}) = \underline{\hspace{2cm}}$ , so our test statistic is  $\underline{\hspace{2cm}}$ .

iii) If  $H_0^A: \sigma_A^2 = 0$  is true, then  $E(\text{MSA}) = rc\sigma_{AB}^2 + rb\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma^2$ , which is *not* the expected value of any of the mean squares above. We will return to this test after discussion of estimates of variance components.

3) *Another model with three random factors:*

$$Y_{ijkl} = \mu + A_i + B_j + C_k + (AB)_{ij} + (BC)_{jk} + \varepsilon_{ijkl}$$

(i.e., no AC or ABC interaction terms), with the appropriate normality and independence conditions.

$$E(\text{MSA}) =$$

$$E(\text{MSB}) =$$

$$E(\text{MSC}) =$$

$$E(\text{MSAB}) =$$

$$E(\text{MSBC}) =$$

$$E(\text{MSE}) =$$

So the tests for AB and BC variance components have denominator  $\underline{\hspace{2cm}}$ .

The test for the C variance component has denominator  $\underline{\hspace{2cm}}$ .

The test for the A variance component has denominator  $\underline{\hspace{2cm}}$ .

The test for the B variance component presents the same problem as noted above.

We can proceed similarly for more factors. There are many possible models. The models are analogous to those for fixed effects, *but* the tests and estimates depend on the expected mean squares, which depend on the model.