TWO OR MORE RANDOM EFFECTS

Example: The factors that influence the breaking strength of a synthetic fiber are being studied. Four production machines and three operators are randomly selected. A two-way factorial experiment is run, with two observations per treatment combination, using raw material from the same production batch, with breaking strength as response.

Here we have two random factors:

- Interest is in the variability of breaking strength over the range of machines and operators.
- Machines and operators for the experiment are randomly chosen.

The *two-way complete model* for two random effects:

Two random factors:
A with a levels
B with b levels

The experimental design is completely randomized, with r_{ij} observations at treatment combination "level i of A and level j of B."

The complete model is:

$$Y_{ijt} = \mu + A_i + B_j + (AB)_{ij} + \varepsilon_{ijt},$$

where:

Each $A_i \sim N(0, \sigma_A^2)$ Each $B_i \sim N(0, \sigma_B^2)$

Each $(AB)_{ij} \sim N(0, \sigma_{AB}^2)$ Each $\varepsilon_{ijt} \sim N(0, \sigma^2)$

The A_i's, B_j's, $(AB)_{ij}$'s, and ε_{ijt}'s are all mutually independent random variables.

Compare and contrast with both the 1-way random effect and 2-way fixed effect complete models.

Note:

1. If all r_{ij}'s have the same value r, then we have a *balanced* design.

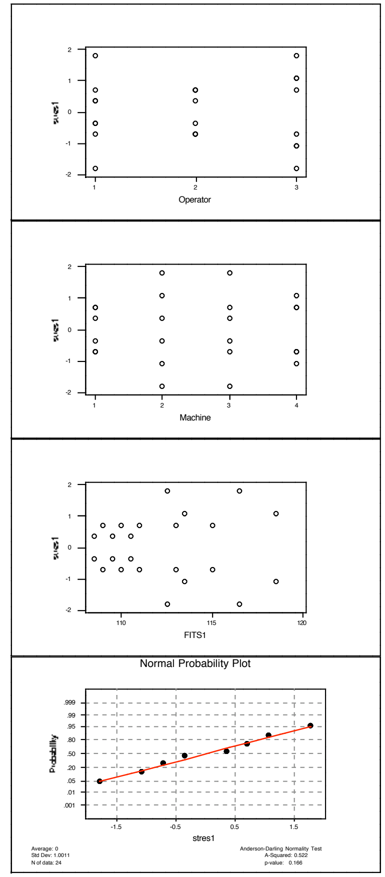
2. If we omit the interaction terms $(AB)_{ij}$, then we obtain the *two-way main effects model for two random effects*.

Least squares gives exactly the same estimates as for the two-way complete fixed effects model.

Thus we can form residuals, sums of squares, and mean squares (in particular, msE, msAB, msA, msB) using the same software routines as for the two-way complete fixed effects model.

In particular, we can use residuals to check some of the model assumptions. As in the one random effect model, normality of the A_i's, B_j's, and $(AB)_{ij}$'s is important – *but we can't really check this with residuals*, since there are few levels of each treatment factor, and the cell averages are *not* independent.

Example: For the fiber strength data, the residual plots and related information that can be used to check the model are:



Max/min standard deviations
By operator 1.55, 2.70
By machine 2.32, 4.46

Calculations using the model assumptions give:

$E(Y_{ijt}) = \mu$

$Var(Y_{ijt}) = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma^2$
(using independence!)

What would the null and alternate hypotheses to test for interaction be?

H_0^{AB}: H_a^{AB}:

We can also consider tests with hypotheses

H_0^A: $\sigma_A^2 = 0$ vs H_a^A: $\sigma_A^2 > 0$

and

H_0^B: $\sigma_B^2 = 0$ vs H_a^B: $\sigma_B^2 > 0$.

But, for example, "A has no effect" is *not* the same as "$\sigma_A^2 = 0$," since A might have an effect through interaction. So "A has no effect" means that *both* σ_A^2 and σ_{AB}^2 are zero.

Recall: For one random effect, the hypothesis test for σ_T^2 used the facts

$E(MSE) = \sigma^2$

and

$E(MST) = c\sigma_T^2 + \sigma^2$.

Analogous calculations (details omitted) for the complete two-way random effects model, assuming a balanced design, yield

$E(MSA) = br\sigma_A^2 + r\sigma_{AB}^2 + \sigma^2$

$E(MSB) = ar\sigma_B^2 + r\sigma_{AB}^2 + \sigma^2$

$E(MSAB) = r\sigma_{AB}^2 + \sigma^2$

$E(MSE) = \sigma^2$.

- Note the pattern of the terms.

- Also note that br is the number of observations at each level of A, and similarly for other coefficients.

To construct test statistics:

For H_0^{AB}:

If H_0^{AB} is true, then E(MSAB) = E(MSE).

If H_0^{AB} is false, then E(MSAB) > E(MSE).

Using the same reasoning as for one random effect,

- MSAB/MSE is an appropriate test statistic.
- It has an F((a-1)(b-1)), ab(r-1)) distribution.

For H_0^{A}:

If H_0^{A} is true, then E(MSA) might *not* equal E(MSE).

But it *does* equal MSAB.

Reasoning similar to the above shows that

- *MSA/MSAB* is a suitable test statistic for H_0^{A}.
- It has an F(a-1, (a-1)(b-1)) distribution.

→ *Note that the test statistic is <u>not</u> the same as in the fixed effects two-way complete model.*

For H_0^{B}: Similar reasoning leads to

MSB/MSAB ~ F(b-1, (a-1)(b-1))

as test statistic.

→ *Again, the test statistic is <u>not</u> the same as in the fixed effects two-way model.*

Comments:

1. The question of whether to test H_0^{A} and H_0^{B} if we reject H_0^{AB} arises as with the two-way compete fixed effects model. But here we have an alternative interpretation:

   We can interpret H_0^{A} as saying the variance component σ_A^2 is zero, rather than saying A has no effect.

2. As with the two-way complete fixed effects model, if we fail to reject H_0^{AB}, we should continue our testing within the complete model, rather than switching to the main effects model for the same data.

Example: Fiber breaking strength – Run the data as both fixed effect and random effect to see the difference in the test statistics.

I. As fixed effect:

Source	DF	SS	MS	F	P
Operator	2	160.333	80.167	21.14	0.000
Machine	3	12.458	4.153	1.10	0.389
Operator*Machine	6	44.667	7.444	1.96	0.151
Error	12	45.500	3.792		
Total	23	262.958			

II. Designating factors as "random"

Source	DF	SS	MS	F	P
Operator	2	160.333	80.167	10.77	0.010
Machine	3	12.458	4.153	0.56	0.662
Operator*Machine	6	44.667	7.444	1.96	0.151
Error	12	45.500	3.792		
Total	23	262.958			

Other models with two or more random effects:

The pattern of expected mean squares continues, and leads us to the appropriate tests.

Examples:

1) Two-way main effects with random factors:

$Y_{ijt} = \mu + A_i + B_j + \varepsilon_{ijt}$, where

each $A_i \sim N(0, \sigma_A^2)$, each $B_i \sim N(0, \sigma_B^2)$,

each $\varepsilon_{ijt} \sim N(0, \sigma^2)$,

and the A_i's, B_j's, and ε_{ijt}'s are all mutually independent random variables.

Expected mean squares:

$E(MSA) = br\sigma_A^2 + \sigma^2$
$E(MSB) = ar\sigma_B^2 + \sigma^2$
$E(MSE) = \sigma^2$.

If H_0^A: $\sigma_A^2 = 0$ is true, then $E(MSA) =$ _________, so our test statistic is _____________.

2) Three way complete random factor model:

$Y_{ijkt} = \mu + A_i + B_j + C_k + (AB)_{ij} + (BC)_{jk} + (AC)_{ik} + (ABC)_{ijk} + \varepsilon_{ijkt}$,

with the appropriate normality and independence conditions.

Expected mean squares:

$E(MSA) = rbc\sigma_A^2 + rc\sigma_{AB}^2 + rb\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma^2$
E(MSB) and E(MSC) are similar

$E(MSAB) = rc\sigma_{AB}^2 + r\sigma_{ABC}^2 + \sigma^2$
E(MSBC) and E(MSAC) are similar

$E(MSABC) = r\sigma_{ABC}^2 + \sigma^2$

$E(MSE) = \sigma^2$.

i) If H_0^{ABC}: $\sigma_{ABC}^2 = 0$ is true, then

E(MSABC) = ___________,

so our test statistic is _____________.

(It has an F-distribution with appropriate degrees of freedom.)

ii) If H_0^{AB}: $\sigma_{AB}^2 = 0$ is true, then

E(MSAB) = __________,

so our test statistic is _______________.

iii) If H_0^{A}: $\sigma_{A}^2 = 0$ is true, then

$E(MSA) = rc\sigma_{AB}^2 + rb\sigma_{AC}^2 + r\sigma_{ABC}^2 + \sigma^2$,

which is *not* the expected value of any of the mean squares above.

→ We will return to this test after discussion of estimates of variance components.

3) Another model with three random factors:

$$Y_{ijkt} = \mu + A_i + B_j + C_k + (AB)_{ij} + (BC)_{jk} + \varepsilon_{ijt}$$

(i.e., no AC or ABC interaction terms),with the appropriate normality and independence conditions.

E(MSA) =
E(MSB) =
E(MSC) =
E(MSAB) =
E(MSBC) =
E(MSE) =

So the tests for the AB and BC variance components have denominator ________________.

The test for the C variance component has denominator _______________.

The test for the A variance component has denominator ______________.

→ The test for the B variance component presents the same problem as noted above.

We can proceed similarly for more factors.

- There are many possible models.
- Model choice depends on the context (prior knowledge).
- The models are analogous to those for fixed effects.
- *However,* the tests and estimates depend on the expected mean squares, which depend on the model.

(We will return to the problem cases after discussing estimating variance components for random factors in the model.)