ANALYSIS OF VARIANCE
FOR THE
COMPLETE TWO-WAY MODEL

The first thing to test in two-way ANOVA:

Is there interaction?

"No interaction" means:

The main effects model would fit.

This in turn means:

In the interaction plot (with A on the horizontal axis and B as marking variable), the corresponding segments of the (piecewise linear) curves for different levels of B are parallel.

More precisely:

For each level i of A and each pair of levels j, q of B, the level j and level q lines in the interaction plot between levels i and i+1 of A are parallel, hence have the same slopes.

In the complete model, these slopes are

$$(\alpha_{i+1} - \alpha_i) + [(\alpha\beta)_{i+1,j} - [(\alpha\beta)_{ij}]$$

and

$$(\alpha_{i+1} - \alpha_i) + [(\alpha\beta)_{i+1,q} - [(\alpha\beta)_{iq}]$$

These are equal if and only if

$$[(\alpha\beta)_{i+1,j} - [(\alpha\beta)_{ij}] - [(\alpha\beta)_{i+1,q} - [(\alpha\beta)_{iq}] = 0.$$

Thus, the null hypotheses

H_0^{AB}: There is no interaction

becomes

H_0^{AB}: $[(\alpha\beta)_{i+1,j} - [(\alpha\beta)_{ij}] - [(\alpha\beta)_{i+1,q} - [(\alpha\beta)_{iq}] = 0$
for all i = 1, 2, … , a - 1
and all unequal j and q from 1 to b

The alternate hypothesis is

H_a^{AB}: $[(\alpha\beta)_{i+1,j} - [(\alpha\beta)_{ij}] - [(\alpha\beta)_{i+1,q} - [(\alpha\beta)_{iq}] \neq 0$
for at least one combination of
i = 1, 2, … , a-1
and unequal j and q from 1 to b

Note: From the equations in H_0^{AB}, we can deduce that

$[(\alpha\beta)_{ij} - (\alpha\beta)_{iq}] - [(\alpha\beta)_{sj} - (\alpha\beta)_{sq}] = 0$
for every combination of i, s from 1 to a
and j, q from 1 to b.

So we could also state the null and alternate hypotheses as

H_0^{AB}: $[(\alpha\beta)_{ij} - (\alpha\beta)_{iq}] - [(\alpha\beta)_{sj} - (\alpha\beta)_{sq}] = 0$
for every combination of i, s from 1 to a
and j, q from 1 to b

and

H_a^{AB}: $[(\alpha\beta)_{ij} - (\alpha\beta)_{iq}] - [(\alpha\beta)_{sj} - (\alpha\beta)_{sq}] \neq 0$
for at least one instance of $i \neq s$, $j \neq q$

For *equal sample sizes*:

Test H_0^{AB} with an F-test testing the submodel (reduced model) determined by H_0^{AB} against the full model:

Compare the sum of squares for error ssE under the full model with the sum of squares for error ssE_0^{AB} under the reduced model.

The difference

$$ssAB = ssE_0^{AB} - ssE$$

is called the *sum of squares for the interaction AB*.

We reject H_0^{AB} in favor of H_a^{AB} when ssAB is large relative to ssE (assuming H_0^{AB} is true). So we'll look at ssAB/ssE.

Full model:

$$Y_{ijt} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijt}$$

Since this is equivalent to the cell-means model, which is a one-way model, we know that

$$\text{ssE} = \sum_i \sum_j \sum_t \hat{e}_{ijt}^{\,2} = \sum_i \sum_j \sum_t \left(y_{ijt} - \bar{y}_{ij\cdot}\right)^2$$

Alternate formulas:

$$\text{ssE} = \sum_i \sum_j \sum_t y_{ijt}^2 - \sum_i \sum_j r_{ij}\bar{y}_{ij\cdot}^{\,2}$$

$$= \sum_i \sum_j \sum_t y_{ijt}^2 - \sum_i \sum_j y_{ij\cdot}^{\,2}/r_{ij}$$

If H_0^{AB} is true, then averaging the equations in H_0^{AB} over s and q gives the equations

$$[(\alpha\beta)_{ij} - (\overline{\alpha\beta})_{i\bullet}] - [(\overline{\alpha\beta})_{\bullet j} - (\overline{\alpha\beta})_{\bullet\bullet}] = 0 \quad \text{for each i, j}$$

So under the reduced model,

$$(\alpha\beta)_{ij} = (\overline{\alpha\beta})_{i\bullet} + (\overline{\alpha\beta})_{\bullet j} - (\overline{\alpha\beta})_{\bullet\bullet}$$

so

$$Y_{ijt} = \mu + \alpha_i + \beta_j + (\overline{\alpha\beta})_{i\bullet} + (\overline{\alpha\beta})_{\bullet j} - (\overline{\alpha\beta})_{\bullet\bullet} + \varepsilon_{ijt}$$

$$= [\mu - (\overline{\alpha\beta})_{\bullet\bullet}] + [\alpha_i + (\overline{\alpha\beta})_{i\bullet}] + [\beta_j + (\overline{\alpha\beta})_{\bullet j}] + \varepsilon_{ijt}$$

$$= \mu^* + \alpha_i^* + \beta_j^* + \varepsilon_{ijt}$$

where

$$\mu^* = [\mu - (\overline{\alpha\beta})_{\bullet\bullet}]$$
$$\alpha_i^* = [\alpha_i + (\overline{\alpha\beta})_{i\bullet}]$$
$$\beta_j^* = [\beta_j + (\overline{\alpha\beta})_{\bullet j}]$$

Thus the reduced model has the *form* of the main effects model, but with different parameters than if we just set interaction terms to zero.

Estimates for the main effects model, assuming equal sample sizes:

Least squares may be used to find estimators of the parameters under the Main Effects Model assumption

$$Y_{ijt} = \mu + \alpha_i + \beta_j + \varepsilon_{ijt}.$$

(See p. 161 of the text for more details.)

For *equal* sample sizes (i.e., *balanced* ANOVA), the resulting normal equations are readily solvable (with added constraints), yielding least squares estimator

$$(*) \quad \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j = \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}$$

for $E[Y_{ijt}] = \mu + \alpha_i + \beta_j$.

Note: 1. Recall that for the complete model, the least squares estimators were

$$\hat{\mu} = \bar{y}_{\cdots}$$
$$\hat{\alpha}_i = \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}$$
$$\hat{\beta}_j = \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots},$$

from which it follows that the least squares estimate for $\mu + \alpha_i + \beta_j$ is the same in both models.

However, in the complete model,

$$E[Y_{ijt}] = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij},$$

which is *not* the same as $E[Y_{ijt}]$ for the main effects model unless $(\alpha\beta)_{ij} = 0$.

2. For *unequal* sample sizes, the normal equations are much messier, so computational solutions are needed. (More later.)

From (*), for the main effects model,

$$\text{ssE} = \sum_i\sum_j\sum_t (\text{y}_{\text{ijt}} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j)^2$$

$$= \sum_i\sum_j\sum_t (\text{y}_{\text{ijt}} - \overline{y}_{i\cdot\cdot} + \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdots})^2,$$

which can be re-expressed as

$$\sum_i\sum_j\sum_t y_{ijt}^2 - br\sum_i \overline{y}_{i\cdot\cdot}^2 - ar\sum_j \overline{y}_{\cdot j\cdot}^2 + abr\overline{y}_{\cdots}^2$$

$$= \sum_i\sum_j\sum_t y_{ijt}^2 - \frac{1}{br}\sum_i y_{i\cdot\cdot}^2 - \frac{1}{ar}\sum_j y_{\cdot j\cdot}^2 + \frac{1}{abr} y_{\cdots}^2$$

Continuing with the test for interaction in the complete two-way model

Applying the above to the reduced model

$$\text{Y}_{\text{ijt}} = \mu^* + \alpha_i^* + \beta_j^* + \varepsilon_{\text{ijt}}$$

in the test for interaction in the complete two-way model, we get (assuming equal sample sizes)

$$\text{ssE}_0^{\text{AB}} = \sum_i\sum_j\sum_t (\text{y}_{\text{ijt}} - \hat{\mu}^* - \hat{\alpha}_i^* - \hat{\beta}_j^*)^2$$

$$= \sum_i\sum_j\sum_t (\text{y}_{\text{ijt}} - \overline{y}_{i\cdot\cdot} + \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdots})^2,$$

which can be re-expressed as

$$\sum_i\sum_j\sum_t (\text{y}_{\text{ijt}} - \overline{y}_{ij\cdot})^2 + \sum_i\sum_j\sum_t (\overline{y}_{ij\cdot} - \overline{y}_{i\cdot\cdot} + \overline{y}_{\cdot j\cdot} - \overline{y}_{\cdots})^2.$$

Since the first term is just ssE for the full model, we have

$ssAB = ssE_0^{AB} - ssE$

$$= \sum_i \sum_j \sum_t (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})^2$$

$$= r \sum_i \sum_j (\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})^2,$$

which can be re-expressed as

$$\frac{1}{r}\sum_i \sum_j y_{ij\cdot}^2 - \frac{1}{br}\sum_i y_{i\cdot\cdot}^2 - \frac{1}{ar}\sum_j y_{\cdot j\cdot}^2 + \frac{1}{abr} y_{\cdots}^2$$

Using the remaining two model assumptions, (that the ε_{ijt} are independent random variables and each $\varepsilon_{ijt} \sim N(0, \sigma^2)$), it can be shown that for the corresponding random variables SSAB and SSE:

When H_0^{AB} is true and sample sizes are equal,

i) $SSAB/\sigma^2 \sim \chi^2((a-1)(b-1))$

ii) $SSE/\sigma^2 \sim \chi^2(n - ab)$

iii) SSAB and SSE are independent.

Thus, when sample sizes are equal and H_0^{AB} is true,

$$\frac{SSAB/(a-1)(b-1)\sigma^2}{SSE/(n-ab)\sigma^2} = \frac{MSAB}{MSE} \sim F((a-1)(b-1), n-ab)$$

Recall: We reject H_0^{AB} in favor of H_a^{AB} when ssAB is large relative to ssE (under the assumption that H_0^{AB} is true). Since msAB/msE is just a constant multiple of ssAB/ssE, we can use msAB/msE as a test statistic, rejecting for large values.

Examples:

1. The battery experiment

2. The reaction time experiment (pp. 98, 148, 157 of textbook). The data are from a pilot experiment to compare the effects of auditory and visual cues on speed of response. The subject was presented with a "stimulus" by computer, and their reaction time to press a key was recorded. The subject was given either an auditory or a visual cue before the stimulus. The experimenters were interested in the effects on the subjects' reaction time of the auditory and visual cues and also in the effect of different times between cue and stimulus. The factor "cue stimulus" had two levels, "auditory" and "visual" (coded as 1 and 2, respectively). The factor "cue time" (time between cue and stimulus) had three levels: 5, 10, and 15 seconds (coded as 1, 2, and 3, respectively). The response (reaction time) was measured in seconds.