MIXED MODELS (Sections 17.7 – 17.8)

Example: Suppose that in the fiber breaking strength example, the four machines used were the only ones of interest, but the interest was over a wide range of operators, and the operators were chosen at random.

This would be an example of a *mixed model*: a model including both fixed and random factors.

The Two-Factor Mixed Model

- There are two factors.
- One (A) is fixed
- The other (B) is random.
- There are two versions of the two-factor model commonly used.
- Both have the same model equation, namely

$$Y_{ijt} = \mu + \alpha_i + B_j + (\alpha B)_{ij} + \varepsilon_{ijt},$$

where

μ and α_i are constants

B_j, $(\alpha B)_{ij}$, and ε_{ijt} are random variables.

- However, the conditions on the random variables differ according to the version of the model.

I) In the ***unrestricted model***, interactions are treated as in the random effects model:

Each $B_i \sim N(0, \sigma_B^2)$

Each $(\alpha B)_{ij} \sim N(0, \sigma_{AB}^2)$

Each $\varepsilon_{ijt} \sim N(0, \sigma^2)$

The B_j's, $(\alpha B)_{ij}$'s, and ε_{it}'s are all mutually independent random variables.

II) The ***restricted model*** requires some intuition building.

To see the idea, start with the cell-means model

$Y_{it} = \mu + \tau_i + \varepsilon_{it}$.

Recall: In order to fit this model by least squares, we need an additional condition. A natural one, if we think of μ as an overall mean and the τ_i's as deviations from it, is

$$\sum \hat{\tau}_i = 0.$$

Indeed, it would be natural to include the condition $\sum \tau_i = 0$ as part of the model, to indicate the interpretation of μ and the τ_i's that we have in mind. (In fact, some people do include this as part of the model.)

The restricted model takes this interpretation for *both* the fixed and mixed interaction effects:

It includes as part of the model the restrictions

$$\sum_{i=1}^{a} \alpha_i = 0 \qquad \text{(i.e., } \alpha_\bullet = 0) \quad \text{and}$$

$$\sum_{i=1}^{a} (\alpha B)_{ij} = 0 \text{ (i.e., } (\alpha B)_{\bullet j} = 0\text{), for each level j of B.}$$

Note that in both cases, the sum is over i – that is, over the *fixed* effects.

So the assumptions for the restricted model, in addition to the model equation, are:

i) $\sum_{i=1}^{a} \alpha_i = 0$

ii) $\sum_{i=1}^{a} (\alpha B)_{ij} = 0$ for each j.

iii) $B_j \sim N(0, \sigma_B^2)$

iv) $\varepsilon_{ijt} \sim N(0, \sigma^2)$

v) $(\alpha B)_{ij} \sim N(0, \frac{(a-1)\sigma_{AB}^2}{a})$

(Note: The messy-looking expression for the variance is just a rescaling convention that makes later formulas less messy.)

vi) The B_j's and ε_{ijt}'s are all mutually independent, *and* independent of the $(\alpha B)_{ij}$'s

vii) The $(\alpha B)_{ij}$'s and $(\alpha B)_{iq}$'s are independent (or at least uncorrelated) for $j \neq q$.

Note: (ii) implies (details left to the student) that

$$\mathrm{Cov}((\alpha B)_{ij}, (\alpha B)_{pj}) = -\sigma_{AB}^2/a \text{ for } i \neq p.$$

Thus the restricted model assumes that, for the same level of B, the interaction effects corresponding to different levels of A are negatively correlated.

Which Model to Use?

- Some people have default preferences, as do most software packages.

- But it's sensible to ask: Does one model fit the situation better than the other? If so, use it!

- One criterion sometimes used: A "limited resource" situation is likely to produce negatively correlated interaction for different A-levels of the same B-level – so the restricted model will reflect this.

Example: Comparing growth of two species of plants -- -so species is a fixed factor with a = 2. Suppose the plot in which the plants are grown is of interest as a random factor. Thus b plots are randomly selected. Each plot is divided into 2r experimental units. Species are randomly assigned to experimental units so that each species is assigned to r experimental units in each plot. (Thus we have a crossed, completely randomized design.) The plants are sown, grown a specified amount of time, harvested, dried, and weighed. Dry weight is the response.

Since resources (nutrients, water, space) in each plot are limited, we expect negative correlations between the final weights of plants in each plot. Thus a restricted model might be better than an unrestricted model here.

Analysis of Mixed Model Experiments

- The basic idea follows along the lines of the methods for developing tests for random effect models.
- But some of the expected mean squares will be different.

An overview of the details:

- We can fit by least squares and use the fits to obtain the mean squares. For example, with a balanced design, we will get SSA = $\frac{1}{br}\sum_i Y_{i\bullet\bullet}^2 - \frac{1}{abr}Y_{\bullet\bullet\bullet}^2$

- Recall that with the one-way random effect model, to calculate E(SSA) (from which we then calculated E(MSA)), we used the equation

  $E(Z^2) = Var(Z) + [E(Z)]^2$

  to calculate $E(Y_{i\bullet}^2)$ and $E(Y_{\bullet\bullet}^2)$.

- We use the same idea here, but the results are messier.

- The reason is that, whereas in the one-way random effects model, $E(Y_{it}) = \mu$, in the mixed effects model, $E(Y_{ijt}) = \mu + \alpha_i$. Consequently, $[E(Y_{i\bullet\bullet})]^2$ and $[E(Y_{\bullet\bullet})]^2$ will involve terms that are quadratic (i.e., degree two polynomials) in the α_i's.

For the *unrestricted* model, we will get:

$E(MSA) = Q(A) + r\sigma_{AB}^2 + \sigma^2$

(where Q(A) denotes a quadratic in the α_i's)

$E(MSB) = ar\sigma_B^2 + r\sigma_{AB}^2 + \sigma^2$

$E(MSAB) = r\sigma_{AB}^2 + \sigma^2$

$E(MSE) = \sigma^2$

->Note that the last three expected mean squares are the same as in the two-way complete random model, but E(MSA) is different.

Hypothesis tests in the unrestricted model:

For fixed effect:

Since A (and no other factor) is fixed, the "main effects" hypotheses for A are

H_0^A: All α_i's are zero vs

H_a^A: At least one $\alpha_i \neq 0$.

It turns out that when H_0^A is true, $Q(A) = 0$, so that

$$E(MSA) = r\sigma_{AB}^2 + \sigma^2 = E(MSAB).$$

The rest of the theory goes over to give

$$MSA/MSAB \sim F(dfA, dfAB)$$

as a suitable test statistic.

For random effects:

The hypotheses corresponding to H_0^B and H_0^{AB} are exactly the same as for the two-way complete random model. Since the expected means squares for B, AB, and E are the same as in that model, the test statistics are also the same for the mixed model as for the random complete model.

For the *restricted* model, the results are similar (but remember that the interpretation of σ_{AB}^2 is different).

These ideas and results generalize to other mixed models.

In particular, E(MS$*$) will be as for the corresponding random effects model, *except that* whenever a fixed effect factor or an interaction involving only fixed effects would occur, there is a quadratic in appropriate fixed effects and their interactions instead of a $\sigma_{..}^2$ term.

Example: For the complete three-way model with two fixed factors A and C and one random factor B, we get

$$E(MSA) = Q(A,AC) + rc\sigma_{AB}^2 + r\sigma_{ABC}^2 + \sigma^2,$$

where Q(A,AC) is a quadratic polynomial in the α_i's and the $(\alpha\gamma)_i$'s.

For details on the expected mean squares, etc., see Section 17.8.2.

-> As with random effects, in some cases there is no suitable single MS for the denominator of the test statistic, so we need to use linear combinations of mean squares and an approximate F test. However, modern software can do much of the detailed calculations for us – we need to focus our attention on selecting an appropriate model and interpreting the output carefully.

Confidence Intervals

Confidence intervals *for contrasts of fixed effects* are calculated as in fixed effects models, *except* that the denominator used in the corresponding hypothesis test (and its degrees of freedom) must be used instead of MSE (and its degrees of freedom).

For example, in a two-way mixed model with fixed factor A, to find confidence intervals for contrasts in the α_i's, use MSAB and dfAB instead of MSE and dfE.

Confidence intervals *for variance components for random effects or interactions involving random effects* are calculated just as for random effects models, but using just the random parts of the model.