SPLIT PLOT DESIGNS

Key Features: More than one type of experimental unit and more than one randomization.

Typical Use: When one factor is difficult to change.

Example (and terminology):

An agricultural researcher is studying the effects of corn variety and irrigation level on corn yields.

Four varieties of corn and three irrigation levels are used.

It's easy to plant a specified variety of corn wherever told to do so. (Thus, variety is an easy-to-change factor.)

However, irrigation is done by large sprinklers that irrigate a large area of land. (Thus, irrigation is a difficult-to-change factor.)

A crossed factorial experiment with just two replications of each of the twelve treatment combinations would require 24 large areas of land.

This is beyond resources available.

To get around the problem:

- Use just six plots of land, chosen so that each can be irrigated at a set level without affecting the irrigation of the others. (These large plots are called the *whole plots*.)

- Randomly assign irrigation levels to each whole plot so that each irrigation level is assigned to exactly two plots. (Irrigation is called the *whole-plot factor*).

- Divide each whole plot into four subplots. (Each subplot is called a *split plot*.)

- Within each whole plot, randomly assign the four corn varieties to the four split plots. (Variety is called the *split-plot factor*.)

1. Draw a picture to illustrate the design.

2. What are the two experimental units and the corresponding two randomizations?

3. How is this design like a two-way crossed factorial design? How are the two designs different?

4. How is this like a randomized complete block design? How are the two designs different?

5. Does the split-plot design introduce any possible confounding?

- *Second example*: An industrial experimenter is studying how the water resistance of wood depends on the pretreatment (two types) and the stain (four types). It turns out to be very difficult to apply the pretreatment to a small wood panel, so instead each type of pretreatment is applied to a whole board, the board is then cut into four smaller wood panels, and one type of stain is applied to each panel. Six whole boards are used.

- Give details to make this into a split-plot design.

- Identify whole and split plots and whole-plot and split-plot factors.

- Is there any confounding?

Note: The book starts with the more complex situation where whole plots are within blocks; we will discuss this generalization later.

Model for Split-Plot Designs

A split-plot experiment can be considered as two experiments superimposed:

- One experiment has the whole-plot factor applied to the large experimental units (whole plots).

- The other experiment has the split-plot factor applied to the smaller experimental units (split plots).

The model will reflect this by including two error terms:

- the whole-plot random error ε^W, and

- the split-plot random error ε^S.

Notation:
- A (with a levels) is the whole-plot factor.

- B (with b levels) is the split-plot factor.

- Each level of A is assigned to l whole plots. (So there are al whole plots. In other words, the number of experimental units in the first randomization is al.)

- Each level of B is assigned to m split plots. (Thus the total number of split plots is bm. In other words, the number of experimental units in the second randomization is bm. In our examples, each level of B is assigned exactly once to each whole plot, so m = al.)

Thus,
- The whole plots are indexed (identified, labeled) by pairs of indices iu, where i = 1, … , a; u = 1, … , l.

- The total collection of split plots is indexed by pairs of indices jt, j = 1, … , b, t = 1, … , m

- Since the total number of split plots can also be calculated as the number of whole plots time the number of split plots per whole plot, we see that the number of split plots per whole plot is (bm)/(al)

Exercise: What are A, B, a, b, l, and m in the examples?

The model equation (Equation 19.2.3 on p. 678) when both A and B are fixed factors is given on two lines to emphasize the two error terms:

$$Y_{iujt} = \mu + \alpha_i + \varepsilon_{iu}^{W}$$

$$+ \beta_j + (\alpha\beta)_{ij} + \varepsilon_{jt(iu)}^{S}$$

where

$$i = 1, \ldots, a \qquad u = 1, \ldots, l$$

$$j = 1, \ldots, b \qquad t = 1, \ldots, m;$$

$$\varepsilon_{iu}^{W} \sim N(0, \sigma_W^2), \ \varepsilon_{jt(iu)}^{S} \sim N(0, \sigma_S^2),$$

ε_{iu}^{W}'s and $\varepsilon_{jt(iu)}^{S}$'s all mutually independent.

Exercise: Why is the nesting notation in the subscript of the split-plot error used?

In our examples (the special case where each level of B is assigned exactly once to each whole plot), the model can be simplified a little. In this special case

- m = al

- if we know i (the level of A) and u (the replication number of the level of A), we have narrowed down to a specific whole plot

- if we also know j (the level of B), we have narrowed down to a specific split plot

Thus we do not need t, and can write the model as

$$Y_{iuj} = \mu + \alpha_i + \varepsilon_{iu}^{W} \\ + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{iuj}^{S}$$

where
$$i = 1, \ldots, a; u = 1, \ldots, l; j = 1, \ldots, b;$$

$$\varepsilon_{iu}^{W} \sim N(0, \sigma_W^2), \varepsilon_{iuj}^{S} \sim N(0, \sigma_S^2),$$

ε_{iu}^{W}'s and ε_{iuj}^{S}'s all mutually independent.

Note: The design and model we are working with are actually the simplest form of split-plot designs. There are many variations. Examples:

- The more general form discussed in the book also has blocks containing the whole plots. (More on this later.)

- Random-effects split-plot designs

- Mixed-effects split-plot designs

- Split-plot designs involving more than two factors.

- Split-split-plot designs, where each split-plot is further divided into subplots.

Analysis of Split-Plot Designs

For now, we will discuss only the two fixed-factor, no block model described above:

$$Y_{iujt} = \mu + \alpha_i + \varepsilon_{iu}{}^W$$

$$+ \beta_j + (\alpha\beta)_{ij} + \varepsilon_{jt(iu)}{}^S$$

where

$i = 1, \dots, a \qquad u = 1, \dots, l$

$\qquad j = 1, \dots, b \qquad t = 1, \dots, m;$

$\varepsilon_{iu}{}^W \sim N(0, \sigma_W{}^2), \varepsilon_{jt(iu)}{}^S \sim N(0, \sigma_S{}^2),$

$\varepsilon_{iu}{}^W$'s and $\varepsilon_{jt(iu)}{}^S$'s all mutually independent.

The model can be fit by least squares.

Model checking: Since there are two error terms in the model, two kinds of residuals need to be checked, the WP (whole plot) and SP (split plot) residuals. To calculate them, first calculate the residuals in the usual way (response values minus fitted values). Then
  - WP residuals are obtained by averaging residuals corresponding to each whole plot.
  - SP (split plot) residuals are obtained by subtracting WP residuals from residuals.

Now make the following model-checking plots:

  - Normal plot of SP residuals (to check normality of split-plot errors)

  - Normal plot of WP residuals (to check normality of whole-plot errors)

  - Plot SP residuals against fitted values (to check constant variance for SP errors)

  - Plot WP residuals against the average fitted value for the corresponding whole plot (to check constant variance for WP errors)

  - Plot WP residuals against SP residuals (to check independence of WP and SP errors)

  - Plot residuals against any other variable available (to check for possible violations of independence).

As suggested by the form of the model, the analysis combines two separate analyses:

- the whole plot analysis, and

- the split-plot analysis.

The idea: Whole plots act like blocks for the split plot analysis.

The *sum of squares for whole plots*, ssW, is calculated in a similar fashion to the sum of squares for blocks in a randomized complete block design.

The *whole plot error sum of squares* is then

$$ssE_W = ssW - ssA.$$

The *split plot error sum of squares* is

$$ssE_S = sstot - ssW - ssB - ssAB.$$

Each sum of squares has an associated degree of freedom.

Mean squares are defined as sums of squares divided by degrees of freedom.

The test statistics are:

Null hypothesis	Test Statistic
H_0^A: No effect of A beyond interaction	msA/msE_W
H_0^B: No effect of B beyond interaction	msB/msE_S
H_0^{AB}: No interaction	$msAB/msE_S$

To run on Minitab and many other programs, use the following trick:

- Create a new variable (usually called W or WP) that indicates which whole plot each observation belongs to. (Use 1, 2, … , al to label the whole plots.)

- In General Linear Model, declare this variable random.

- In specifying factors, indicate that this factor is nested in A (the whole plot factor).

Example: In the experiment studying the effect of pretreatment and stain on water resistance, the data (including W) are as shown:

pretreat	stain	resist	W
2	2	53.5	4
2	4	32.5	4
2	1	46.6	4
2	3	35.4	4
2	4	44.6	5
2	1	52.2	5
2	3	45.9	5
2	2	48.3	5
1	3	40.8	1
1	1	43.0	1
1	2	51.8	1
1	4	45.5	1

1	2	60.9	2
1	4	55.3	2
1	3	51.1	2
1	1	57.4	2
2	1	32.1	6
2	4	30.1	6
2	2	34.4	6
2	3	32.2	6
1	1	52.8	3
1	3	51.7	3
1	4	55.3	3
1	2	59.2	3

In Minitab, use General Linear Model.

    Response: resist

    Model: pretreat W(pretreat) stain pretreat* stain

    Random: W

The output is:

General Linear Model: resist versus pretreat, stain, W

```
Factor        Type Levels Values
pretreat      fixed      2 1 2
W(pretreat)   random     6 1 2 3 4 5 6
stain         fixed      4 1 2 3 4

Analysis of Variance for resist, using Adjusted SS for Tests

Source           DF    Seq SS    Adj SS    Adj MS      F      P
pretreat          1    782.04    782.04    782.04   4.03  0.115
W(pretreat)       4    775.36    775.36    193.84  15.25  0.000
stain             3    266.00    266.00     88.67   6.98  0.006
pretreat*stain    3     62.79     62.79     20.93   1.65  0.231
Error            12    152.52    152.52     12.71
Total            23   2038.72
```

Note:

1. Ignore the P-value for W.

2. This does not work with Minitab 10.

3. ssE_W is in the line W(pretreat).

4. ssE_S is in the line Error

5. Check that the sums of squares add as indicated above.

6. Check that the test ratios are as they should be.

7. Note that ssE_W is much larger than ssE_S. This is typical. Why?

Interpret the output:

Note: If we don't designate W as random, we get different output:

General Linear Model: resist versus pretreat, stain, W

```
Factor        Type Levels Values
pretreat      fixed     2 1 2
W(pretreat)   fixed     6 1 2 3 4 5 6
stain         fixed     4 1 2 3 4

Analysis of Variance for resist, using Adjusted SS for Tests

Source          DF    Seq SS    Adj SS    Adj MS      F      P
pretreat         1    782.04    782.04    782.04  61.53  0.000
W(pretreat)      4    775.36    775.36    193.84  15.25  0.000
stain            3    266.00    266.00     88.67   6.98  0.006
pretreat*stain   3     62.79     62.79     20.93   1.65  0.231
Error           12    152.52    152.52     12.71
Total           23   2038.72
```

What is different? How do we know the first method is the one we want?

Split Plot Designs with Blocks

The split plot model we have discussed is a special case (namely, just one block) of a more general split plot design, where the whole plots are themselves nested within blocks.

If the randomization is such that each level of A appears exactly once per block and each level of B appears exactly once per whole plot, the model for this design can be expressed as

(Equation 19.2.2 in Dean-Voss)

$$Y_{hij} = \mu + \theta_h + \alpha_i + \varepsilon_{i(h)}^{\ W}$$

$$+ \beta_j + (\alpha\beta)_{ij} + \varepsilon_{j(hi)}^{\ S}$$

where
$$h = 1, \ldots, s; \ i = 1, \ldots, a; \ j = 1, \ldots, b$$

$$\varepsilon_{i(h)}^{\ W} \sim N(0, \sigma_W^2), \ \varepsilon_{j(hi)}^{\ S} \sim N(0, \sigma_S^2),$$

$\varepsilon_{i(h)}^{\ W}$'s and $\varepsilon_{j(hi)}^{\ S}$'s all mutually independent.

(θ_h is the effect of the hth block.)

Sums of squares are as for the case previously discussed, *except* now

$$ssE_W = ssW - ss\theta - ssA.$$

The appropriate test statistics are just as before:

Null hypothesis	Test Statistic
H_0^A: No effect of A beyond interaction	msA/msE_W
H_0^B: No effect of B beyond interaction	msB/msE_S
H_0^{AB}: No interaction	$msAB/msE_S$

Example: (The oats example on p. 681)

- The experimental area was divided into s = 6 blocks.

- Each block was subdivided into a = 3 whole plots.

- 3 varieties of oats (factor A) were sown on whole plots according to a randomized complete block design (so every variety appeared in every block exactly once).

- Each whole plot was further divided into b = 4 split plots.

- 4 levels of manure were applied to the split plots according to a randomized complete block design (so each level of B appeared in each whole plot exactly once).

To run on Minitab, enter for model:

   BLOCK A WP(BLOCK) B A*B

Specify BLOCK as random. (Minitab then automatically specifies WP as random, since it is nested in BLOCK. In some older versions, if you try to specify WP as random, you may get an error message.)

The output is:

General Linear Model: Y versus BLOCK, A, B, WP

```
Factor      Type Levels Values
BLOCK     random   6   1 2 3 4 5 6
A          fixed   3   0 1 2
WP(BLOCK) random  18   1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
17 18
B          fixed   4   0 1 2 3

Analysis of Variance for Y, using Adjusted SS for Tests

Source      Model DF   Reduced DF     Seq SS
BLOCK         5           5          15875.3
A             2           2           1786.4
WP(BLOCK)    12          10+          6013.3
B             3           3          20020.5
A*B           6           6            321.8
Error        43          45           7968.8
Total        71          71          51985.9

+ Rank deficiency due to empty cells, unbalanced nesting,collinearity,
or an undeclared covariate.
  No storage of results or further analysis will be done.
```

The last comment essentially means that we can't make residual plots and need to do the remaining calculations by hand.

The "reduced degrees of freedom" are what we need.

Calculating test statistics:

For interaction:

$$msAB/msE_S = (ssAB/6)/(ssE_S/45)$$

$$= (321.8/6)/(7968.8/45)$$

$$= 53.63/177.08 = 0.30$$

For an $F(6, 45)$ distribution, this gives p-value

$$1 - 0.0664 = .9336.$$

This is consistent with no interaction.

For level of manure (factor B):

$$msB/ msE_S = (ssAB/3)/(ssE_S/45)$$

$$= (20020.5/3)/(7968.8/45)$$

$$= 6673.5/177.08 = 37.69$$

For an $F(3,45)$ distribution, this gives p-value

$$1 - 1.0000 = 0.0000 \text{ (to four decimal places)}$$

This gives strong evidence that the level of manure makes a difference.

For variety of oats (factor A):

$$msA/msE_W = (ssA/2)/(ssWP(BLOCK)/10)$$

$$= (1786.4/2)/(6013.3/10)$$

$$= 893.18/601.33 = 1.49.$$

For an $F(2,10)$ distribution, this gives p-value

$$1 - 0.7286 = .2714.$$

This is consistent with no effect of variety.

A further analysis would involve contrasts;

Note that $msE_S = 177.08$ (the estimate of error variance for whole plots) is noticeably smaller than $msE_W = 601.33$ (the estimate of error variance for split plots), as is typical of split plot designs.