CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS (Section 17.3.5)

These play the role for random effects that confidence intervals for contrasts play for fixed effects.

Confidence intervals for σ^2: Constructed just as for fixed effects; see Section 3.4.6 or the class notes *Choosing Sample Sizes*.

Confidence intervals involving σ_T^2:

Three types of are of interest:

- For σ_T^2
- For σ_T^2/σ^2
- For $\sigma_T^2/(\sigma_T^2 + \sigma^2)$

The first cannot be done exactly, so we'll take that last.

Confidence intervals for σ_T^2/σ^2:

We use the fact (see notes *Testing for Treatment Effect as a Proportion of Error Variance)* that

$$\frac{MST/(c\sigma_T^2 + \sigma^2)}{MSE/(\sigma^2)} \sim F(v-1, n-v),$$

where c is a certain constant defined in terms of n,v, and the r_i; c = r if the design is balanced) (See notes *Random Effects Models* or Section 17.3)

If we want a $(1-\alpha)100\%$ CI for σ_T^2/σ^2, take

$f_1 = F(v-1, n-v, 1-\alpha/2)$
(so that there is area $\alpha/2$ to the left of f_1 in the F(v-1, n-v) distribution), and

$f_2 = F(v-1, n-v, \alpha/2)$
(so that there is area $\alpha/2$ to the right of f_2 in the F(v-1, n-v) distribution).

[Picture!]

Then

$$\text{Prob}\left(f_1 \le \frac{MST/(c\sigma_T^2+\sigma^2)}{MSE/(\sigma^2)} \le f_2\right) = 1-\alpha$$

Equivalently:

$$\text{Prob}\left(f_1 \le [MST/MSE]\,[\boldsymbol{\sigma}^2/(c\boldsymbol{\sigma}_T^2 + \boldsymbol{\sigma}^2)] \le f_2\right)$$
$$= 1-\alpha,$$

Working with the left inequality:

$$(c\boldsymbol{\sigma}_T^2 + \boldsymbol{\sigma}^2)/\boldsymbol{\sigma}^2 \le (MST/MSE)(1/f_1)$$

$$c(\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2) + 1 \le (MST/MSE)(1/f_1)$$

$$c(\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2) \le (MST/MSE)(1/f_1) - 1$$

Working with the right inequality: is equivalent to

$$(MST/MSE)(1/f_2) \le (c\boldsymbol{\sigma}_T^2 + \boldsymbol{\sigma}^2)/\boldsymbol{\sigma}^2$$

$$= c(\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2) + 1$$

$$(MST/MSE)(1/f_2) - 1 \le c(\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2)$$

So

$$\text{Prob}\left((1/c)[\,(MST/MSE)(1/f_2) - 1] \le \boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2 \le (1/c)[\,(MST/MSE)(1/f_1) - 1]\right) = 1 - \alpha.$$

Thus desired confidence interval has left endpoint

$$(1/c)[\,(msT/msE)(1/f_2) - 1]$$

and right endpoint

$$(1/c)\,(msT/msE)(1/f_1) - 1)$$

Here, "(a, b) is a 95% confidence interval for $\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2$"
Means:

Note: Conceivably the left hand endpoint could be less than 0, which is unrealistic. If it is < 0, do *not* give in to the temptation to replace it by zero; that would give the false impression of a smaller confidence interval than warranted.

Example: Use the loom data to find a 95% confidence interval for $\boldsymbol{\sigma}_T^2/\boldsymbol{\sigma}^2$.

Confidence intervals for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ *= the proportion of the total variance if the response attributable to the treatment level*:

Such confidence intervals are readily obtained from confidence intervals for σ_T^2/σ^2 as follows.

Divide both numerator and denominator of $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ by σ^2 to obtain

$$\sigma_T^2/(\sigma_T^2 + \sigma^2) = \frac{\sigma_T^2/\sigma^2}{\left(\sigma_T^2/\sigma^2\right)+1} = f\left(\sigma_T^2/\sigma^2\right),$$

$$\text{where } f(x) = x/(x + 1)) = \frac{1}{1+\frac{1}{x}}.$$

From the last formula for f(x), we can see that f(x) is an increasing function of x. i.e., $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ is an increasing function of σ_T^2/σ^2.

Thus if (a,b) is a (1-α)100% confidence interval for σ_T^2/σ^2, then (f(a), f(b)) = (a/(a + 1), b/(b + 1)) is a (1-α)100% confidence interval for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$.

Note: $\sigma_T^2/(\sigma_T^2 + \sigma^2)$ is sometimes called the "population intraclass correlation coefficient"

Caution: The phrase "intraclass correlation coefficient" is also used to refer to other things.

Example: With the loom data, find a 95% confidence interval for $\sigma_T^2/(\sigma_T^2 + \sigma^2)$.

Confidence intervals for $\boldsymbol{\sigma_T^2}$: There is no exact method. There are several approximate methods. Here's one. It's useful if $\boldsymbol{\sigma_T^2}$ is not too small, and is adaptable to more complicated models.

Recall that U = (1/c)(MST – MSE) is an unbiased estimator of $\boldsymbol{\sigma_T^2}$. If we knew its distribution, we could use that to get confidence intervals for $\boldsymbol{\sigma_T^2}$ in the usual way.

However, U does not have a tractable distribution.

But it *is* true that

$$U/\boldsymbol{\sigma_T^2} \approx \chi^2(x)/x, \text{ where}$$

$$x \approx \frac{(msT - msE)^2}{\frac{(msT)^2}{(v-1)} + \frac{(msE)^2}{(n-v)}}$$

(Note: This formula is given correctly on p. 605 of the text, but incorrectly on p. 600.)

x from this formula is not usually an integer, so we need to interpret degrees of freedom in the χ^2 distribution as the parameter in a formula for the pdf. (This is analogous to the two-sample, unequal variance t-test.)

Thus (Draw a picture!)

$$P(\ \chi^2(x, 1-\alpha/2) \prec xU/\boldsymbol{\sigma_T^2} \prec \chi^2(x, \alpha/2)\ ) \approx 1-\alpha,$$

where $\prec$ means "is less than or approximately equal to", and $\chi^2(x,\beta)$ is the value with proportion β of the $\chi^2(x)$ distribution to its *right*.

The left and right approximate inequalities are, respectively, equivalent to

$$\boldsymbol{\sigma_T^2} \prec xU/\chi^2(x, 1-\alpha/2) \quad \text{and}$$

$$\boldsymbol{\sigma_T^2} \succ xU/\chi^2(x, \alpha/2).$$

Thus if
u = (1/c)(msT – msE)
(which is our estimate for $\boldsymbol{\sigma_T^2}$),

then

$$(xu/\chi^2(x, \alpha/2),\ xu/\chi^2(x, 1-\alpha/2))$$

is an approximate (1-α)100% confidence interval for $\boldsymbol{\sigma_T^2}$.

<u>Example</u>: With the loom data, find a 95% confidence interval for $\boldsymbol{\sigma_T^2}$.