# ANALYSIS OF COVARIANCE (Chapter 9)

Recall from the handout Randomized Complete Designs:

*Nuisance factor*: A factor that is expected to have an effect on the response, but is not a factor of interest for the purpose of the experiment.

Types of nuisance factors and how to deal with them in designing an experiment:

| **Characteristics** | **Examples** | **How to treat** |
|---|---|---|
| Unknown, uncontrollable | Experimenter bias, effect of order of treatments | Randomization |
| Known, uncontrollable, measurable | IQ, weight, previous learning | Analysis of Covariance |
| Known, moderately controllable (by choosing rather than adjusting) | Temperature, location, time, batch, particular machine or operator, age, gender, order, IQ, weight | Blocking |

We will now discuss Analysis of Covariance, to deal with nuisance factors that can be measured, but cannot be controlled or cannot be measured in advance. (Measuring in advance would allow the possibility of either blocking or analysis of covariance.)

*Example*: Recall the balloon experiment (p. 62. #12) from the February 15 assignment: Color of balloon was the only factor of interest; inflation time was the response. In part (d) of that problem, you were asked to plot the data for each color in the order in which it was collected. The following graph shows a plot of inflation time vs order:


There is a clear trend of decreasing inflation time as order increases. This is plausible: the experimenter probably got better at blowing up the balloons with practice, leading to the decrease. The decrease means that the observations are not independent, so the independence assumption of ANOVA is not met, and hence the conclusions of the ANOVA analysis are not valid.

However, since the dependence of time on order appears to be approximately linear (on average), and there is no reason to suspect that the dependence of time on order should differ for different colors, the data appear to fit a *one-way analysis of covariance model*:

**One-way Analysis of Covariance Model**

The model assumptions are:

- A completely randomized experiment is used to compare the effects of the levels of a single treatment factor T on the response variable Y. There is also a nuisance factor (covariate) X whose value can be measured before or during the experiment.
- $Y_{it} = \mu^* + \tau_i + \beta x_{it} + \varepsilon_{it}$ (1)
  *Note that this says that there is a linear relationship between E(Y) and x, with the same slope for each level of the treatment factor.*
- $\varepsilon_{it} \sim N(0, \sigma^2)$
- $\varepsilon_{it}$'s mutually independent,
- $x_{it}$ is not affected by the treatment

where $x_{it}$ is the value of X for observation t of level i ($i = 1, 2, \ldots, v$; $t = 1, 2, \ldots, r_i$) and $\mu^*$, $\tau_i$, and $\beta$ are constants.

It is more common to use the following form of the model equation:

$Y_{it} = \mu + \tau_i + \beta(x_{it} - \bar{x}_{\bullet\bullet}) + \varepsilon_{it}$ (2)

This is obtained from equation (1) by letting $\mu = \mu^* + \beta\bar{x}_{\bullet\bullet}$. In (1), $\mu^* + \tau_i$ is the mean response when $x_{it} = 0$; in (2), $\mu + \tau_i$ is the mean response when $x_{it} = \bar{x}_{\bullet\bullet}$.

*Extensions of the basic model*:

- The treatment factor can be as in the cell-means model for an experiment with several crossed factors.
- Random and/or nested factors can be used; mixed models are possible.
- More than one covariate can be included in the model.
- Quadratic, etc. terms in x may be used to model more complex relationships between Y and X.
- Interaction involving covariates can be modeled.

Of course, the analysis gets more complex as the model becomes more complex.

*Fitting and model checking*

Least squares can be used to fit the model.

*Fitting with Minitab:*

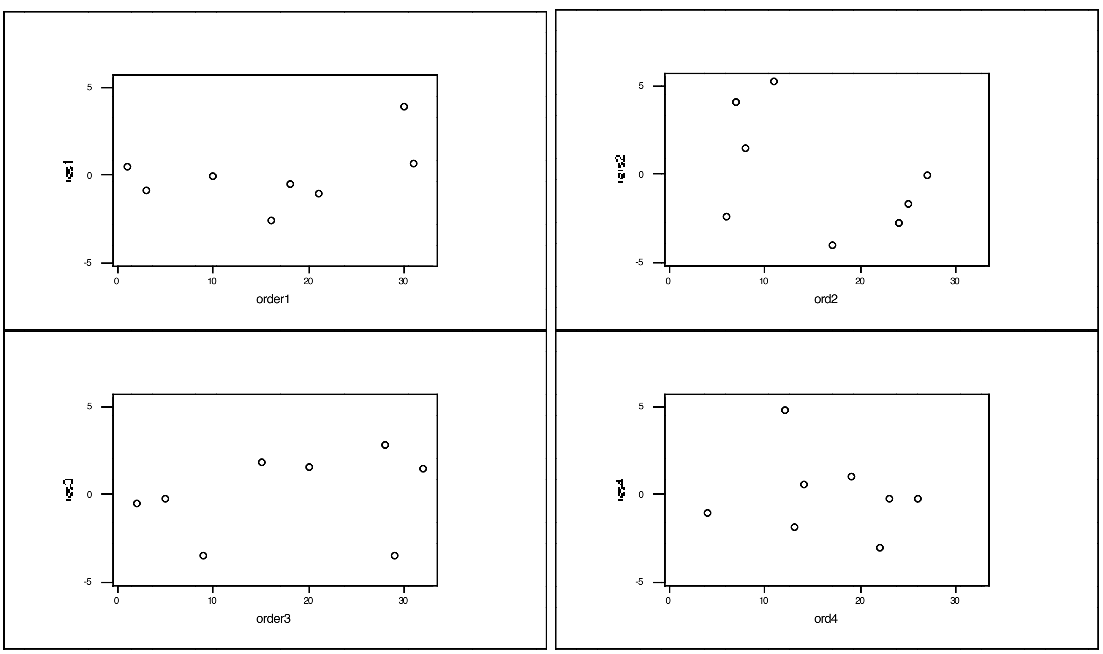
- Use General Linear Model under ANOVA on the Stat menu.
- Enter just the factor of interest under "Model" and the covariate under "Covariates" (You may have to click on "covariates" to get the box to enter the covariate in)
- Save fits and residuals as with ANOVA

Residuals can then be used to check the model assumptions:

- First plot residuals vs covariate, for each treatment level, *using the same scale*.
    - If any one of the plots appears *non-linear*, the model does not fit. A single plot marked by treatment level can sometimes, but not always, accomplish this; to make separate graphs in Minitab, you first need to unstuck by the treatment factor, then mark “same X and Y” under “multiple graphs” when making graphs.
    - If the model fits, each the plot for each level should have a random pattern, with no indication of non-linear trend or non-horizontal linear trend. (Remember that the model assumption is that the linearity relationship is between E(Y) and x, not between Y and x.)
    - A non-linear pattern indicates that the dependence on covariate is not linear; a linear but not horizontal pattern indicates that a model with an interaction term (to allow different slopes for different levels) is needed. (Note: The discussion in the book is not correct.)
- If the model passes the above checks, proceed with other model checks as for ANOVA.

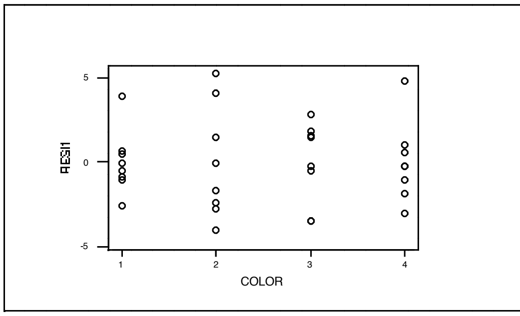
*Model checking plots for Balloon example, using the above ANCOVA model:*

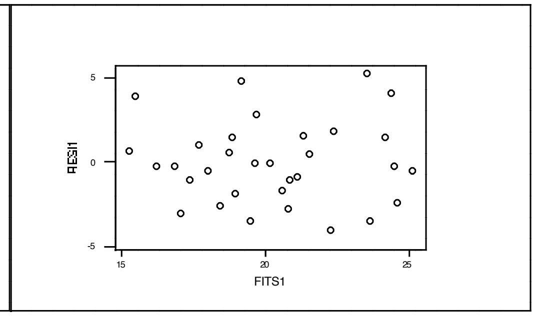
Plots of residuals vs covariate for each color, on the same scale:



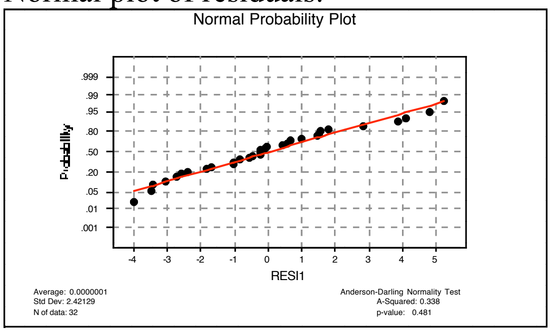

Eight points per plot don’t give definitive information, but there is no clear sign of non-linearity; all plots are consistent with a random pattern about the horizontal, as we would expect if the model fits.

Plots of residuals vs color and vs fits:




Normal plot of residuals:




Note: The Minitab ouput pointed out two observations with large standardized residuals:

| Obs. | TIME | Fit | Stdev.Fit | Residual | St.Resid |
|---|---|---|---|---|---|
| 11 | 28.8000 | 23.5510 | 0.9458 | 5.2490 | 2.17R |
| 12 | 24.0000 | 19.1635 | 0.9458 | 4.8365 | 2.00R |

But neither of these is surprising in light of our sample size.

*Analysis*:

There are two hypothesis tests:

1) $H_0^T$: All $\tau_i$'s are equal
   (i.e., the levels of T have the same effect, after accounting for X)

   vs

   $H_a^T$: At lest two $\tau_i$'s are different
   (i.e., the levels of T have different effects, after accounting for X).

2) $H_0^\beta$: $\beta = 0$ (no linear dependence on the covariate) vs $H_a^\beta$: $\beta \neq 0$

Each test is developed by the familiar idea of comparing the error sum of squares under the full model with the error sum of squares under the reduced model obtained by assuming the null hypothesis is true. These differences are called the *sum of squares for treatment and for slope* – but since each of these sums of squares is calculated assuming the other parameters are fixed, we use the condition notation:

Sum of squares for treatment = ss(T|β) =
(error sum of squares under the reduced model assuming $H_0^T$ is true) – ssE

Sum of squares for slope = ss(β|T) =
(error sum of squares under the reduced model assuming $H_0^{\beta}$ is true) – ssE.

The corresponding random variables are called SS(T|β) and SS(β|T). Each has an associated degrees of freedom. Mean squares are obtained by dividing by the degrees of freedom. The test statistics are:

For $H_0^T$: MS(T|β)/MSE ~ F(1, n-v-1)

For $H_0^{\beta}$: MS(β|T)/MSE ~ F(1, n-v-1)

*Balloon example:* The Minitab output is

```
Analysis of Variance for TIME

Source     DF    Seq SS    Adj SS    Adj MS      F      P
ORDER       1   120.818   120.835   120.835  17.95  0.000
COLOR       3   127.679   127.679    42.560   6.32  0.002
Error      27   181.742   181.742     6.731
Total      31   430.239
```

The first F-test tells us:

The second F-test tells us:

*One-way ANCOVA with interaction*:

Recall that the one-way ANCOVA model given above assumes that there is a linear relationship between E(Y) and the covariate, with the same slope for each level of the treatment factor. By adding an interaction term, we can allow for different slopes for different values of the covariate. The model equation can be given as

$$Y_{it} = \mu + \tau_i + \beta_i(x_{it} - \bar{x}_{\bullet\bullet}) + \varepsilon_{it} \qquad (3)$$

The model can be fit in Minitab by specifying model A|X (or A X A*X), still listing X as covariate.

Using this model for the balloon data gives output

```
Source        DF    Seq SS    Adj SS    Adj MS      F      P
COLOR          3   127.661    65.999    22.000   3.36  0.035
ORDER          1   120.835   132.225   132.225  20.21  0.000
COLOR*ORDER    3    24.710    24.710     8.237   1.26  0.311
Error         24   157.032   157.032     6.543
Total         31   430.239
```

The third test statistic indicates no evidence of interaction – that is, no evidence of different slopes for different colors, as we suspected.

*Treatment Contrasts and Confidence Intervals*

A contrast $\sum_i c_i \tau_i$ (where $\sum_i c_i = 0$) is estimable with estimate $\sum_i c_i (\bar{y}_{i\bullet} - \hat{\beta}\bar{x}_{i\bullet})$. See p. 286 of the textbook for details on estimated variance and confidence intervals.

Unfortunately, the Tukey, Dunnett, and Hsu methods for multiple comparisons do not work for ANCOVA. However, the Sheffe and Bonferroni methods still apply. (See p. 287 for examples with the balloon data.)