# CONDITIONAL MEANS AND VARIANCES, CONTINUED: CONDITIONAL VARIANCES

**Marginal Variance**: The *definition* of the (population) (marginal) variance of a random variable Y is

$$\mathrm{Var}(Y) = E([Y - E(Y)]^2 )$$

What does this say in words (and pictures)?

There is another *formula* for Var(Y) that is sometimes useful in computing variances or proving things about them. It can be obtained by multiplying out the squared expression in the definition:

$$\mathrm{Var}(Y) = E([Y - E(Y)]^2 ) = E(Y^2 - 2YE(Y) + [E(Y)]^2)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(Fill in details, and say the final result in words!)

**Conditional Variance**: Similarly, if we are considering a conditional distribution Y|X, we define the *conditional variance*

$$\mathrm{Var}(Y|X) = \text{Variance of the conditional distribution } Y|X$$

$$= E([Y - E(Y|X)]^2 \mid X)$$

(Note that *both* expected values here are conditional expected values.)

What does this say in words (and pictures)?

Exercise: Derive another formula for the conditional variance, analogous to the second formula for the marginal variance. (And say it in words!)

**Conditional Variance as a Random Variable**: As with E(Y|X), we can consider Var(Y|X) as a random variable. For example, if Y = height and X = sex for persons in a certain population, then Var(height | sex) is the variable which assigns to each person in the population the variance of height for that person's sex.

**Expected Value of the Conditional Variance**: Since Var(Y|X) is a random variable, we can talk about its expected value. Using the formula $\mathrm{Var}(Y|X) = E(Y^2|X) - [E(Y|X)]^2$, we have

$$E(\mathrm{Var}(Y|X)) = E(E(Y^2|X)) - E([E(Y|X)]^2)$$

We have already seen that the expected value of the conditional expectation of a random variable is the expected value of the original random variable, so applying this to $Y^2$ gives

$$(*) \qquad E(Var(Y|X)) = E(Y^2) - E([E(Y|X)]^2)$$

**Variance of the Conditional Expected Value**: For what comes next, we will need to consider the variance of the conditional expected value. Using the second formula for variance, we have

$$Var(E(Y|X)) = E([E(Y|X)]^2) - [E(E(Y|X))]^2$$

Since $E(E(Y|X)) = E(Y)$, this gives

$$(**) Var(E(Y|X)) = E([E(Y|X)]^2) - [E(Y)]^2.$$

**Putting It Together**:

Note that (*) and (**) both contain the term $E([E(Y|X)]^2)$, but with opposite signs. So adding them gives:

$$E(Var(Y|X)) + Var(E(Y|X)) = E(Y^2) - [E(Y)]^2,$$

which is just Var(Y). In other words,

$$(***) \quad Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$$

In words: The marginal variance is the sum of the expected value of the conditional variance and the variance of the conditional means.

**Consequences**:

I) This says that two things contribute to the marginal (overall) variance: the expected value of the conditional variance, and the variance of the conditional means. (See Exercise) Moreover, $Var(Y) = E(Var(Y|X))$ if and only if $Var(E(Y|X)) = 0$. What would this say about $E(Y|X)$?

II) Since variances are always non-negative, (***) implies

$$Var(Y) \geq E(Var(Y|X)).$$

III) Since $Var(Y|X) \geq 0$, $E(Var(Y|X))$ must also be $\geq 0$. (Why?). Thus (***) implies

$$Var(Y) \geq Var(E(Y|X)).$$

Moreover, $Var(Y) = Var(E(Y|X))$ if and only if $E(Var(Y|X)) = 0$. What would this imply about $Var(Y|X)$ and about the relationship between Y and X?

IV) Another perspective on (***) (cf. Textbook, pp. 36 - 37):

Note that:

i) $E(Var(Y|X)$ is a weighted average of $Var(Y|X)$

ii) $Var(E(Y|X) = E([E(Y|X) - E(E(Y|X))]^2)$

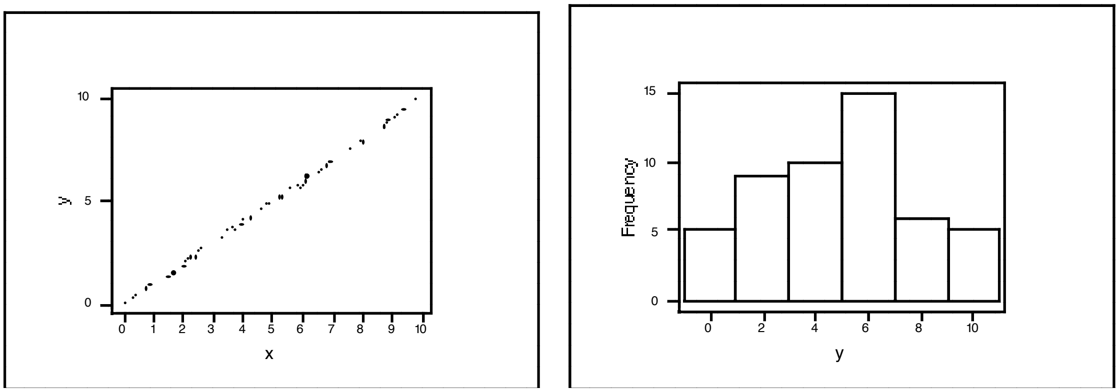
$= E([E(Y|X) - (E(Y)]^2)$,

which is a weighted average of $[E(Y|X) - (E(Y)]^2$

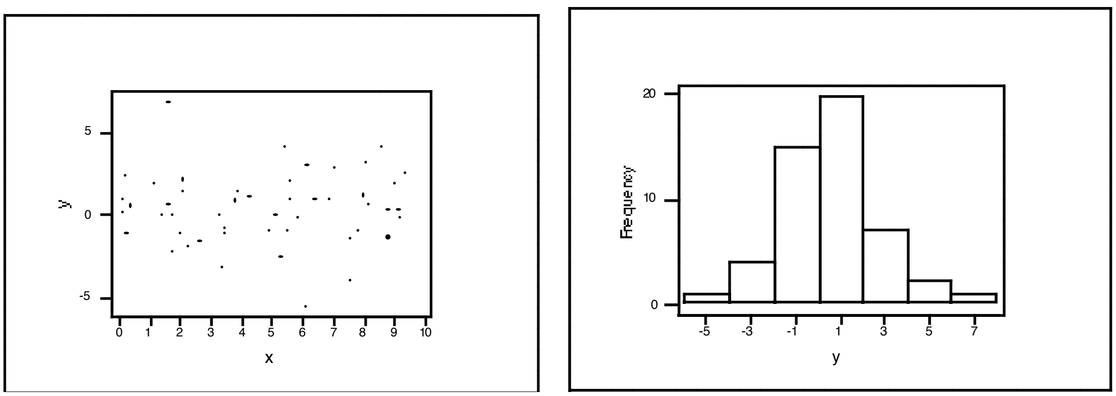
Thus, (***) says that $Var(Y)$ is a weighted mean of $Var(Y|X)$ plus a weighted mean of $[E(Y|X) - (E(Y)]^2$ (and is a weighted mean of $Var(Y|X)$ if and only if all conditional expected values $E(Y|X)$ are equal to the marginal expected value $E(Y)$.)

EXERCISE: What contributes most to Var(Y): Var(E(Y|X)) or E(Var(Y|X))?

A.




B.




C.