PREDICTION INTERVALS

What if we want to estimate Y|x not just E(Y|x)?
The only estimator available is

$$\hat{y} = \hat{\eta}_0 + \hat{\eta}_1 x \text{ - -}$$

the same estimator we used for E(Y|x), calling it $\hat{\text{E}}$(Y|x).

Intuitively, we should not be able to estimate Y as closely as E(Y|x): Estimating E(Y|x) involves *sampling error* only, but estimating Y|x must take into account the *natural variability of the distribution Y|x* as well as sampling error.

[Try drawing a picture to illustrate this.]

The increased variability in estimating Y as compared to estimating E(Y|x) requires us to use a different standard error.

To help avoid confusion, estimating Y is called *prediction*. (Unfortunately, this produces possible new confusion: sometimes people think that regression prediction must involve the future, or that it is exact.) Similarly, the estimate is sometimes called y_{pred} rather than $\hat{y}$ (so $y_{pred} = \hat{\eta}_0 + \hat{\eta}_1 x$), and the associated error is called *prediction error*:

Prediction error: For a *new* (or additional) observation y chosen from Y|x independently of $y_1, \dots, y_n$, we define

Prediction error = y - $\hat{\text{E}}$(Y|x) (= y - $\hat{y}$)

- Draw a picture
- Compare and contrast with the error e|x and the residuals $\hat{e}_i$
- Prediction error is a random variable -- its value depends on the choice of $y_1, \dots, y_n$, and y

For fixed x,

E(prediction error) = E(Y|x - $\hat{\text{E}}$(Y|x)) = ________________________________

Also,

Var(prediction error) = Var(Y|x - $\hat{\text{E}}$(Y|x)| $x_1, \dots, x_n$)

= Var(Y|x, $x_1, \dots, x_n$) + Var($\hat{\text{E}}$(Y|x)| $x_1, \dots, x_n$)) (Why?)

= Var(Y|x) + Var($\hat{\text{E}}$(Y|x)| $x_1, \dots, x_n$))

= σ^2 + Var($\hat{\text{E}}$(Y|x)) for short

$$= \sigma^2 + \sigma^2\left(\frac{1}{n} + \frac{(x-\bar{x})^2}{SXX}\right)$$

$$\sigma^2\left(1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{SXX}\right)$$

Define: se(y_{pred}|x) = $\hat{\sigma}\sqrt{1+\frac{1}{n}+\frac{(x-\bar{x})^2}{SXX}}$

$$= \sqrt{\hat{\sigma}^2 + V\hat{a}r(\hat{E}(Y \mid x))}$$

Sampling Distribution of Prediction Error:

- $\hat{\mathrm{E}}$(Y|x) is a linear combination of the y_i's ⇒ y|x - $\hat{\mathrm{E}}$(Y|x) is a linear combination of y and the y_i's .
- This plus independence and normality assumptions on y|x and the yi's ⇒ y|x - $\hat{\mathrm{E}}$(Y|x) is normally distributed.
- It can be shown that this implies that

$$\frac{Y \mid x - \hat{E}(Y \mid x)}{se(y_{pred} \mid x)} \sim \text{t(n-2)}.$$

Thus we can use this statistic to calculate a *prediction interval* (or "confidence interval for prediction") for y.

Recall: A 90% *confidence* interval for the conditional mean E(Y|x) is an interval produced by a process which, for 90% of all independent random samples y_1, … , y_n taken from Y|x_1, … , Y|x_n, respectively, yields an interval containing the parameter E(Y|x) (assuming all model assumptions fit).

Compare and contrast: A 90% *prediction* interval (or "confidence interval for prediction") is an interval produced by a process which, for 90% of all independent random samples y_1, … , y_n, y taken from Y|x_1, … , Y|x_n , Y|x, respectively, yields an interval containing the new sampled value y (assuming all the model assumptions fit).

Thus the prediction interval is *not* a confidence interval in the usual sense -- since it is used to estimate a value of a random variable rather than a parameter.