BIVARIATE NORMAL DISTRIBUTIONS M348G/384G

Random variables X_1 and X_2 are said to have a *bivariate normal distribution* if their joint pdf has the form

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} \right]$$

(Here, $\exp(u) = e^u$.)

- Compare and contrast with the pdf of the univariate normal:

  $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right]$$

- The five parameters completely determine the distribution (if it is known to be bivariate normal).
- There are lots of bivariate normal distributions
- The pdf is symmetric (suitably interpreted) in the two variables.

Properties: (Calculations left to the interested student)

1. $X_1 \sim N(\mu_1, \sigma_1)$ (What calculation needed?)
2. $X_2 \sim N(\mu_2, \sigma_2)$ (What calculation needed?)
3. $\rho = \rho_{X_1, X_2}$ (What calculation needed?)

Note:
- If you know that a distribution is bivariate normal, and know its marginal distributions, do you know the joint distribution?
- A bivariate distribution might have both marginals normal, but not be bivariate normal.

  Example: X and Z independent standard normal.

  $$Y = \begin{cases} Z \text{ if } XZ > 0 \\ -Z \text{ if } XZ < 0 \end{cases}$$

  Try sketching a sample from the bivariate distribution of X and Y.

One way bivariate normals arise:

Theorem: If X and Y are independent normal random variables and if X_1 and X_2 are each linear combinations of X and Y (e.g., if $X_1 = X$ and $X_2 = Y$), then X_1 and X_2 are bivariate normal.

Consequence: By the Central Limit Theorem and empirical observation, (approximate) normals occur often in nature -- hence also (approximate) bivariate normals.

Also: Many jointly distributed variables can be transformed to (approximately) bivariate normal.

Standard bivariate normal: $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$.

- So marginals are ___________________
- Any ρ between -1 and 1 is possible.
- So different standard bivariate normals have the same marginals.

Uncorrelated bivariate normals: $\rho = 0$ implies:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}{2}\right]$$

$$= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right]\exp\left[-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

$$= f_{X_1}(x_1) f_{X_2}(x_2),$$

which implies ________________________________

Thus: Bivariate normal plus uncorrelated implies __________________

Contours: In the special case of uncorrelated variables:

$f(x_1, x_2) = c$ (constant) says

$$\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 = k \quad [= -2\ln(2\pi\sigma_1\sigma_2), \text{ also a constant}],$$

which describes ____.

If also the joint distribution is *standard* normal, then the contour lines are ___________.

Will this happen any other time?

If ρ ≠ 0 (still assuming bivariate normality), then (details left to the interested student) the contours will have equations of the form

$$k = \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2.$$

The contours are ____
Special case of standard normal (other cases can be obtained by translating and scaling these):

$$k = x^2 - 2\rho xy + y^2$$

If ρ = 0, these are __________________________________.

If ρ ≠ 0, these are ellipses tilted at a 45° angle to the coordinate axes, with lengths

$\sqrt{\frac{k}{2(1-\rho)}}$ in the SW-NE direction

$\sqrt{\frac{k}{2(1+\rho)}}$ in the NW-SE direction.

(This is not obvious!)

Thus:

If ρ is close to 1, the ellipse is long in the ____________ direction.
If ρ is close to -1, the ellipse is long in the ____________ direction.