JOINT, MARGINAL AND CONDITIONAL DISTRIBUTIONS

Joint and Marginal Distributions: Suppose the random variables X and Y have joint probability density function (pdf) $f_{X,Y}(x,y)$. The value of the cumulative distribution function $F_Y(y)$ of Y at c is then

$$
\begin{aligned}
F_Y(c) &= P(Y \le c) \\
&= P(-\infty < X < \infty,\ Y \le c) \\
&= \text{the volume under the graph of } f_{X,Y}(x,y) \text{ above the region ("half plane")}
\end{aligned}
$$

R: $\begin{cases} -\infty < x < \infty \\ y \le c \end{cases}$ (Sketch the region and volume yourself!)

Setting up the integral to give this area, we get

$$
\begin{aligned}
F_Y(c) &= \iint_R f_{X,Y}(x,y)\,dx\,dy \\
&= \int_{-\infty}^{c} \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \right) dy \\
&= \int_{-\infty}^{c} g(y)\,dy,
\end{aligned}
$$

where $g(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$.

Thus the pdf of Y is $f_Y(y) = F_Y'(y) = g(y)$.

In other words, the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Similarly, the marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

In words: The marginal pdf of X is ____

Note: When X or Y is discrete, the corresponding integral becomes a sum.

Joint and Conditional Distributions:

First consider the case when X and Y are both discrete. Then the marginal pdf's (or pmf's = probability mass functions, if you prefer this terminology for discrete random variables) are defined by

$$f_Y(y) = P(Y = y) \qquad \text{and} \quad f_X(x) = P(X = x).$$

The joint pdf is, similarly,

$$f_{X,Y}(x,y) = P(X = x \text{ and } Y = y).$$

The conditional pdf of the conditional distribution Y|X is

$$\begin{aligned} f_{Y|X}(y|x) &= P(Y = y|X = x) \\ &= \frac{P(X = x \text{ and } Y = y)}{P(X = x)} \\ &= \frac{f_{X,Y}(x,y)}{f_X(x)}. \end{aligned}$$

In words:

Is this also true for continuous X and Y? In other words:

Does $\int_c^d \frac{f_{X,Y}(a,y)}{f_X(a)}\,dy = P(c \le Y \le d \mid X = a)$ for every a, c, and d?

It is enough to show that $\int_{-\infty}^d \frac{f_{X,Y}(a,y)}{f_X(a)}\,dy = P(Y \le d \mid X = a)$ for every a and d. (Draw a picture to help see why!).

Starting with the right side, we can reason as follows:

(Draw pictures to help see the steps!)

$P(Y \le d \mid X = a) \approx P(Y \le d \mid a \le X \le a + \Delta x)$ (for small Δx)

$$\begin{aligned} &= \frac{P(Y \le d \text{ and } a \le X \le a + \Delta x)}{P(a \le X \le a + \Delta x)} \\ &\approx \frac{P(Y \le d \text{ and } a \le X \le a + \Delta x)}{f_X(a)\Delta x} \\ &= \frac{\int_{-\infty}^d \left(\int_a^{a+\Delta x} f_{X,Y}(x,y)\,dx\right) dy}{f_X(a)\Delta x} \end{aligned}$$

$$\approx \frac{\int_{-\infty}^{d} f_{X,Y}(a,y)\Delta x\, dy}{f_X(a)\Delta x}$$

$$= \frac{\int_{-\infty}^{d} f_{X,Y}(a,y)\, dy}{f_X(a)}$$

$$= \int_{-\infty}^{d} \frac{f_{X,Y}(a,y)}{f_X(a)} dy \text{, as desired.}$$

Summarizing: The conditional distribution Y|X has pdf

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

In word equations:

$$\textit{Conditional density of Y given X} = \frac{\textit{joint density of X and Y}}{\textit{marginal density of X}}$$

(and, of course, the symmetric equation holds for the conditional distribution of X given Y).