WEIGHTED MEANS AND MEANS AS WEIGHTED SUMS

In the Speeds Problem we saw that there is more than one kind of "average." In this handout, we will explore this topic further.

The ordinary mean is sometimes called the "*arithmetic mean*" to distinguish it from other types of means.

Note on pronunciation: When "arithmetic" is used as an adjective (as in "arithmetic mean"), it is pronounced "air-rith-MAT-ic" or "air-rith-MET-ic" -- i.e., accent on the third syllable. (Analogy: "geometric").

The most common way to think of the average (arithmetic mean) of numbers is to add them up and divide by the total number of summands:

e.g., the average of 1,1,2,3, 4,4,4 is (1+1+2+3+4+4+4)/7

But we could write this two other ways:

1. "Distributing" the denominator gives

$$(1/7)1 + (1/7)1 + (1/7)2 + (1/7)3 + (1/7)4 + (1/7)4 + (1/7)4.$$

Thus we have the mean as a sum of coefficients times the original numbers in the list. *Note that the sum of the coefficients is 1*.

2. Collecting like terms gives

$$(2x1 + 2 + 3 + 3x4)/7 = (2/7)1 + (1/7)2 + (1/7)3 + (3/7)4.$$

Now we have a sum of coefficients times the *distinct* values (not allowing repetitions) in the original list of numbers. The coefficient of a value is the *proportion* of that value in the original list of numbers. We still have the coefficients adding to 1, but they are no longer all the same. We now see the mean as a *weighted sum* of the *distinct values*, where each value is weighted according to its proportion in the total list of numbers. This perspective prompts two generalizations of the arithmetic mean.

A. **Weighted Means**

To form a *weighted mean* of numbers, we first multiply each number by a number ("weight") for that number, then add up all the weighted numbers, then divide by the sum of the weights. We often do this in computing course grades – e.g., weighting the final exam twice as much as a midterm exam. The ordinary (arithmetic) mean is a weighted mean with all weights equal to 1.

Another way to describe a *weighted mean* of a list of numbers is a sum of coefficients times the numbers, where the coefficients add up to 1. In this case, the coefficients are called the *weights*. (*Note the ambiguity in the use of "weight".)* If all the weights are the same, we get the ordinary arithmetic mean.

Why are these two descriptions equivalent?

Examples of weighted means:
1. The discussion above shows that the ordinary (arithmetic) mean can also be considered as a *weighted mean* of the *distinct values* being averaged, with the weight of a value being its proportion in the original list of numbers being averaged.

2. In part (b) of the Average Speeds Problem (Problem 2 in the handout "What Do You Mean by Average?"), the average speed can be written as a weighted mean:

$$\text{Average speed} = \frac{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n}{a_1 + a_2 + \ldots + a_n} =$$

$$\frac{a_1}{a_1 + a_2 + \ldots + a_n} v_1 + \frac{a_2}{a_1 + a_2 + \ldots + a_n} v_2 + \ldots + \frac{a_n}{a_1 + a_2 + \ldots + a_n} v_n$$

$$= w_1 v_1 + w_2 v_2 + \ldots + w_n v_n,$$

where $w_i = \dfrac{a_i}{a_1 + a_2 + \ldots + a_n}$

Note that the sum of the w_i's is 1. Thus, the answer to part b in the Average Speeds Problem can be seen as *a weighted mean of the original speeds, with the weight of each speed being the fraction (proportion) of the total number of intervals that are traveled at that speed.*

3. Another place where weighted means are important is when the purpose of the study is to compare means of two groups, but the two groups are appreciably different in size. Consider for example, a study whose purpose is to compare the educational and workforce experiences of male and female electrical engineers. There are many fewer women in electrical engineering than men, so a simple random sample of all engineers in the population would include very few women, and therefore not give as good estimates for the women as the men. Instead, the researchers would use a *stratified* sample – they might, for example, sample 200 men and 200 women. But then if they want a *sample* "average" that estimates the average for *all* electrical engineers, including both men and women, they need to take a weighted average.

Exercise: Suppose that the total number of men in the *population* being studied (e.g., *all* electrical engineers) is N_M and the total number of women in the *population* is N_w. If 200 of each sex are sampled, what would be appropriate weights for calculating an average of some variable (e.g., salary; number of years in the profession) on which data are collected, if the intent is to estimate the average for the *entire population* of interest (e.g., all electrical engineers)?

B. **Means of Random Variables** Viewing the mean of a list of (not necessarily distinct) numbers (e.g., exam scores) as a weighted mean of the distinct values occurring in the list prompts us to define the *mean of a discrete numerical random variable* as

Mean of X = $\sum f_X(x)x$,

where the sum is over all values that X can take on.

Example: You put two more dots on the "one" side of a fair die to make it into a "three". X is the random variable "number that comes up when you roll the die." Then the mean of X is (1/6)2 + (2/6)3 + (1/6)4 + (1/6)5 + (1/6)6

We can extend this idea to continuous random variables by using an integral instead of a sum: The *mean of a continuous random variable* as

Mean of X = $\int_{-\infty}^{\infty} x f_X(x)dx$

The mean of a random variable is also called the *expected value* or the *expectation* of the random variable, and denoted E(X).

Problems:

1. a. Guess the mean of the discrete random variable X that only takes on values a and b, with P(X =a) = p and P(X = b) = 1-p.
   b. Now calculate the mean of this random variable.
   c. How far is the mean from a? From b? Does this make sense?
   d. If a = 1 and b = 0, X is called a *Bernoulli* random variable. So if X is Bernoulli, then its mean is E(X) = _____. (*Note*: For a Bernoulli random variable, cases where X = 1 are often called "successes, so p is often called the "probability of success")

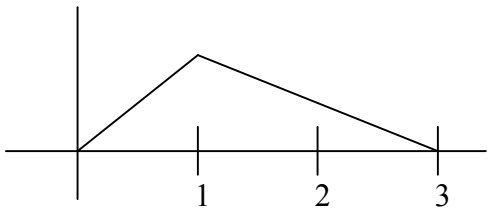
In Problems 2 and 3, for the random variable described,

a. *First* guess what the mean is.
b. *Then* calculate the mean.

2. X has the uniform distribution on [A,B].

3. The pdf of X has the graph shown:

(First you'll need to find a formula for the pdf.)



C. **Harmonic Means and Weighted Means**

i. In part (c) of the Average Speeds Problem, you showed that if you travel a certain distance at speed v_1, then the same distance at speed v_2, and so forth, finishing by traveling that same distance at speed v_n, your average speed for the whole trip is

$$v_{av} = \frac{n}{\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}}$$

The expression on the right is called the *harmonic mean* of $v_1, v_2, \ldots, v_n$. The harmonic mean of a set of numbers can be described as: The reciprocal of the arithmetic mean of the reciprocals of the numbers. Usually it is just defined for positive numbers. (WHY?)

The name "harmonic" presumably comes from the fact that in the harmonic series 1 + ½ + 1/3 + ¼ + …, each term is the harmonic mean of the preceding and the following term. (Check it out algebraically!). The harmonic series is so named because it is connected to what is called the "harmonic series" in music – this refers to the fact that a string can make sounds corresponding to lengths that go into it an integer number of times. (See http://www.philtulga.com/harmonics.html for a demonstration.)

The harmonic mean also occurs in other situations involving rates, and in one method of Congressional Apportionment. (See "7. Dean's Method" at http://mathdl.maa.org/mathDL/46/?pa=content&sa=viewDocument&nodeId=3163)

ii. Your answer in part (d) of the average speeds problem can be called a *weighted harmonic mean*. (How?)

iii. The harmonic mean itself is a weighted arithmetic mean:

Harmonic mean of $v_1, v_2, \ldots, v_n = w_1v_1 + w_2v_2 + \ldots + w_nv_n$,

where

$$w_i = \frac{\frac{1}{v_i}}{\frac{1}{v_1} + \frac{1}{v_2} + \ldots + \frac{1}{v_n}}$$ (Check out the algebra yourself!)

Problems:

II. a. Calculate and compare the arithmetic and harmonic means of some numbers in several cases and form a conjecture as to whether the harmonic mean is always less than, always greater than, or sometimes less than and sometimes greater than the arithmetic mean.

b. Prove your conjecture in the case n = 2 (that is, means of just two numbers). [Note: The general case is harder; we may come back and do it later.]

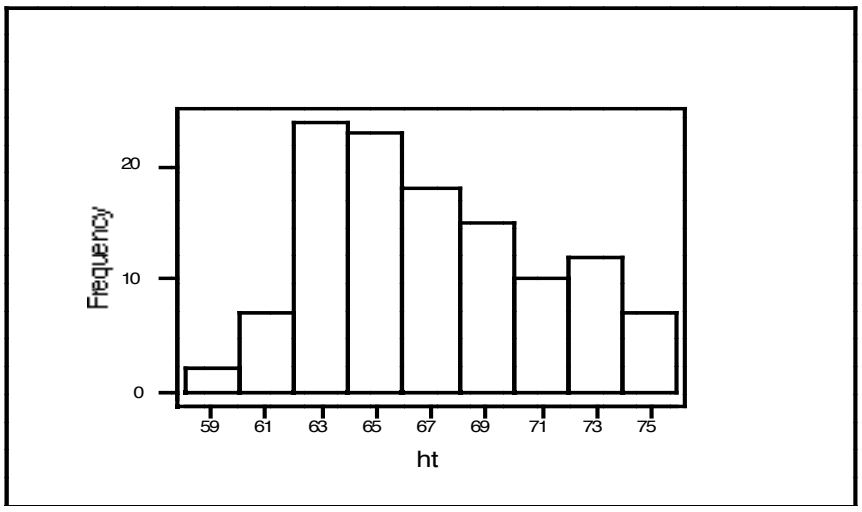
III. The arithmetic and harmonic means can both be considered "measures of center." The median is another measure of center. Recall that the *median* of a set of numbers is the number in the middle when the numbers are listed in order (or the average of the two middle numbers if the number is even.) So the median of the numbers 1,1,2,3, 4,4,4 is 3 (and would still be 3 if the numbers were rearranged), and the median of the numbers 1,1,2,3, 4,4 is (2+3)/2 = 2.5 (and would be the same if the numbers were rearranged). So the median has equal numbers (of the numbers in the original list, counting each one as many times as it occurred in the original list) on either side of it. This suggests how to define the *median of a continuous random variable*: The number with equal probability of the random variable occurring on either side of that number. That is, the median M is a number with the property

$P(X<M) = P(X>M)$.

a. Use a symmetry argument to find the median of the uniform and normal distributions.

b. Figure out the median of distribution #3 in part I.

c. Estimate the median of the empirical distribution whose histogram is shown:



D. **More Means**

The harmonic mean is the reciprocal of the average of the reciprocals of the numbers involved. More succinctly: HM = $\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right)^{-1}$ Looking at it this way, we can see that both the harmonic mean and the arithmetic mean are special cases of the *generalized mean with exponent p*:

$M_p(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{p}\right)^{1/p}$

The arithmetic mean is the case p = ___, and the harmonic mean is the case p = ____. Another special case, often called the *root mean square*, is the case p = 2:

$$\text{RMS} = \text{RMS}(x_1, \ldots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

(The square of RMS is sometimes called the *mean square.*)

The RMS formula might remind you of the formula for the *population standard deviation* of a finite population $x_1, \ldots, x_n$ (that is, of a discrete random variable whose list of possible values is $x_1, \ldots, x_n$, with each values listed in proportion to its probability.) Using σ for population standard deviation,

$$\sigma = \text{RMS}(x_1 - \bar{x}, \ldots, x_n - \bar{x}) = \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right)^{1/2}$$

(Note: This is not quite the same as the *sample standard deviation, which uses n-1 instead of n in the denominator).* One of the problems below will explore the connection between the RMS, the arithmetic mean, and the population standard deviation.

Problems:

IV. a. Use the appropriate formulas and algebra to prove the following relationship:

$$\text{RMS}^2 = \text{AM}^2 + \sigma^2,$$

where RMS is the root mean square of $x_1, \ldots, x_n$, AM is their arithmetic mean (also know as $\bar{x}$), and σ is their standard deviation, as defined above. [Hint: Start with the formula for σ^2. Multiply out $(x_i - \bar{x})^2$, then regroup the addition ("distribute the summation sign") so that you have summations of similar terms. Then use the fact that $\sum_{i=1}^{n} x_i = n\bar{x}$. Things should drop out nicely.]

b. Use part (a) to conclude that RMS is always $\leq$ AM. When are they equal?