What Is Probability?

The idea: Uncertainty can often be "quantified"

- i.e., we can talk about *degrees* of certainty or uncertainty.
- This is the idea of probability: a higher probability expresses a higher degree of certainty/a lower degree of uncertainty that something will happen.
- Statistical inference techniques are based on probability.

Dictionary definition:

- American Heritage Dictionary Definition 3: "*Math*. A number expressing the likelihood of occurrence of a specific event, such as the ratio of the number of experimental results that would produce the event to the total number of results considered possible."
- AHD Definition 1 of Likelihood: "The state of being likely or probable; probability."
- A circular definition -- not very helpful!

But compare:

- What is time?
- What is a point?

Differing Perspectives on Probability

Some confusions involving probability and statistics involve confusing three perspectives on probability:

- Classical (also called "a priori" or "theoretical")
- Empirical (also called "a posteriori" or "frequentist" or, confusingly "classical")
- Subjective (also called "personal")

Terminology: The things we consider the probabilities of are called ***events***.

Examples:

- The event that the number showing on a die we have rolled is 5.
- The event that it will rain tomorrow.
- The event that someone in a certain group will contract a certain disease within the next five years.

Classical ("a Priori" or "Theoretical") Perspective

- Situation: a non-deterministic process ("random process") with n *equally likely* outcomes.
- e.g., toss a fair die: Six equally likely outcomes,
- P(A) ("the probability of event A") is defined to be m/n, where A is satisfied by exactly m of the n outcomes
- e.g., toss a fair die; A = "an odd number comes up" ~~> P(A) = 3/6.

Pros and Cons of Classical Probability

Pros

- Conceptually simple for many situations.

Cons

- Doesn't apply when outcomes are not equally likely.
- Doesn't apply when there are infinitely many potential outcomes

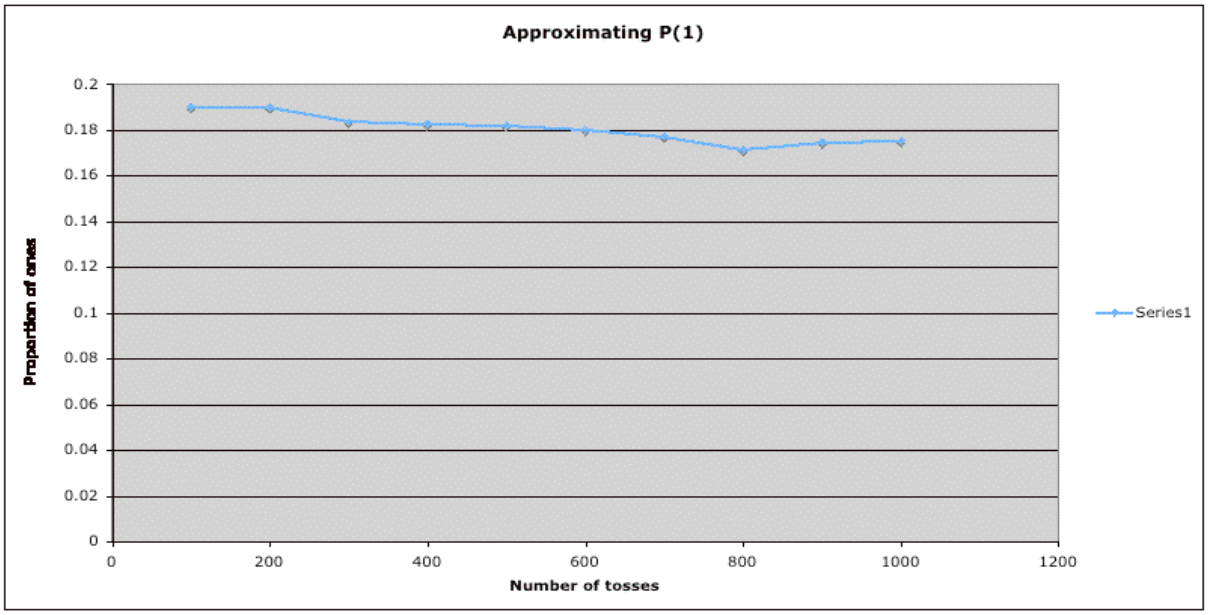
Empirical ("A Posteriori" or "Frequentist" or "Classical") Perspective

- Consider a process that we can imagine performing repeatedly (e.g., tossing a die); we consider an event A that can be described in terms of the results of the process (e.g., "the number that comes up is less than 4")
- P(A) is defined to be the limiting value, as we perform the process more and more times, of the ratio

$$\frac{\textit{Number of times A occurs}}{\textit{Number of times process is repeated}}$$

- E.g., toss a fair die; A = six lands up
- E.g., toss a die that is suspected of *not* being fair; A = six lands up.

Illustration of the Empirical Perspective of Probability: The graph shows the results of a simulation of tossing a die 1200 times, recording after each toss the proportion of times "1" comes up on the (simulated) die.



- The horizontal axis shows the number of tosses of a fair die.
- The vertical axis shows the proportion of those tosses that came up 1.
- The trend of the graph is that as the number of tosses increases, the proportion of ones approaches the true probability of 1/6 = 0.16666... .
- Notice that the zeroing in on the true value is not steady -- in this particular simulation, there is some moving upward from 800 to 1000.
- If we increased the number of tosses to 2000, 3000, etc., we would expect the calculated proportions to vary less and less from the true value.

Pros and Cons of Empirical Probability

Pros

- Covers more cases than classical.
- Intuitively, agrees with classical when classical applies.

Cons

- Repeating the identical experiment an infinite number of times (sometimes even twice) is physically impossible.
- How many times must we perform the process to get a good approximation to the limiting value?

The empirical view of probability is the one that is used in most commonly used statistical inference procedures, including the ones covered in this course. These are called frequentist statistics.

Subjective Perspective ("Personal Probability")

- An individual's personal measure of belief that the event will occur.
- e.g., P(the stock market will go up tomorrow).
- Needs to be "coherent" to be workable.
    - e.g., P(stock market goes up tomorrow) = .6 and P(stock market goes down tomorrow) = .7 are inconsistent.

Pros and Cons of Subjective Probability

Pros

- Applicable in situations where other definitions are not.
- Fits intuitive sense of probability.
- Can be considered to extend classical and empirical views.

Cons

- Can vary from individual to individual.
- Requires "coherence" conditions; are people always that rational?

The subjective perspective of probability fits well with Bayesian statistics, which are an alternative to the more common frequentist statistical methods. (Bayesian statistics will not be covered in this course.)

Unifying Perspective: Axiomatic Model of Probability

- The coherence conditions needed for subjective probability can be proved to hold for the classical and empirical definitions.
- The axiomatic perspective codifies these coherence conditions, so can be used with any of the above three perspectives.

The axiomatic perspective:

A function P from events to non-negative numbers satisfying:

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$ (S = certain event; sample space)
3. P(union of mutually exclusive events) = sum of P of individual events