Differential and Integral Equations Volume 15, Number 1, January 2002, Pages 115–128

FEEDBACK NULL CONTROLLABILITY OF THE SEMILINEAR HEAT EQUATION

Mihai Sîrbu
Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA 15213, USA

(Submitted by: Viorel Barbu)

Abstract. In this paper we study the null controllability of the semilinear heat equation studying the properties of the minimum energy that one needs to steer an initial state x in 0. We prove this is locally Lipschitz, and consequently we obtain the expected optimal feedback law. We also characterize the value function as the unique positive viscosity solution (of the corresponding Hamilton–Jacobi equation with singular final data) which tends to 0 on admissible trajectories, or as the minimal positive viscosity supersolution.

1. Introduction

This work is concerned with the feedback null controllability of the state system

$$\begin{cases} y_t(x,t) - \triangle y(x,t) + f(y(x,t)) = m(x)u(x,t) \text{ for}(x,t) \in Q = \Omega \times (0,T) \\ y(x,t) = 0 \text{ for } (x,t) \in \Sigma = \partial\Omega \times (0,T) \\ y(x,0) = y_0(x) \text{ for } x \in \Omega \end{cases} \tag{1.1}$$

where Ω is an open and bounded subset of R^n with a smooth boundary $\partial\Omega$ and m is the characteristic function of an open subset $\omega \Subset \Omega$. Here $\triangle$ is the Laplace operator with respect to x. We assume that $f \in C^1(R),\ f(0) = 0$ and $|f'(r)| \leq L(\forall) r \in R$. So f is globally Lipschitz.

In the sequel we want to rewrite the state system (1.1) in a semigroup form. For this purpose we define $H := L^2(\Omega)$ and $A : D(A) \subset H \to H$ to be $Au = -\triangle u$, where $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. We also define the Nemytskii operator associated to f, $F : H \to H$ by $(Fu)(x) = f(u(x))$ for almost every $x \in \Omega$ and the bounded linear control operator $B : H \to H$ by $(Bu)(x) = m(x)u(x)$ for almost every $x \in \Omega$.

Accepted for publication: October 2000.
AMS Subject Classifications: 49N35, 93B05, 49L25.

Taking into account all this notation, the state system becomes

$$\begin{cases} y' + Ay + Fy = Bu \text{ on } (0,T) \\ y(0) = y_0. \end{cases} \tag{1.2}$$

We know that $-A$ generates a C_0-semigroup, F is globally Lipschitz on H and B is bounded. So, for each $u \in L^2(0,T;H)$ and $y_0 \in H$, (1.2) has a unique mild solution $y \in C([0,T];H)$. For the definition of mild solution, see [7].

With these hypotheses we know from [4] that (1.1) (or (1.2)) is globally null controllable; i.e., for each $x \in H$ and $0 \leq t < T$ there exists $u \in L^2(t,T;H)$ such that

$$\begin{cases} y' + Ay + Fy = Bu \text{ on } (t,T) \\ y(t) = x \text{ and } y(T) = 0. \end{cases} \tag{1.3}$$

For fixed $x \in H$ we want to find a feedback law for $u^* \in L^2(t,T;H)$ which fulfills (1.3) and minimizes the energy

$$J(u) := \frac{1}{2} \int_t^T |u|^2 \, d\tau. \tag{1.4}$$

Throughout the paper we will denote by $|\cdot|$ the norm in any Hilbert space, and by $\|\cdot\|_X$ the norm of a Banach space in case we want to emphasize the space X. We want to study the minimization problem $\min J(u)$ subject to (1.3) by means of dynamic programming arguments. For this reason we define the value function $\varphi : [0,T) \times H \to R$ by

$$\varphi(t,x) = \inf\left\{ \int_t^T \tfrac{1}{2}|u|^2 d\tau : y' + Ay + Fy = Bu,\ y(t) = x,\ y(T) = 0 \right\}. \tag{1.5}$$

We expect that φ is the solution (unique if possible), in a weak sense, of the Hamilton–Jacobi equation

$$\varphi_t - \langle (Ax + Fx), \nabla\varphi \rangle - \tfrac{1}{2}|B^*\nabla\varphi|^2 = 0 \tag{1.6}$$

subject to the formal final condition

$$\varphi(T,x) = \begin{cases} +\infty & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{1.7}$$

In Section 2 we prove that for fixed $0 \leq t < T$, $\varphi(t,\cdot)$ is locally Lipschitz, and consequently we have the optimal feedback law,

$$u^*(s) \in -B^*\partial\varphi(s, y^*(s)), \tag{1.8}$$

where by $\partial\varphi$ we denote the generalized gradient of φ with respect to x in the sense of F. Clarke.

In Section 3 we characterize the value function as the minimal positive viscosity supersolution of (1.6) subject to (1.7) or as the unique positive viscosity solution which tends to 0 on admissible trajectories. In particular, the existence of a positive viscosity supersolution is equivalent to the null controllability of the state system.

2. Properties of the value function. Feedback laws

Before stating the main result of this section, we need some preliminaries. First, we would like to remind the reader of some properties of the operators A and F: A is self-adjoint and $-A$ generates a compact C_0-semigroup, F is Gateaux differentiable and $(F'(u)v)(x) = f'(u(x))v(x)$ for almost every $x \in \Omega$. In general, F is not Fréchet differentiable.

Then we define, for each $\varepsilon > 0$, $\varphi_\varepsilon : [0,T] \times H \to R$, by

$$\varphi_\varepsilon(t,x) = \inf\Big\{ \int_t^T \frac{1}{2}|u|^2 d\tau + \frac{1}{2\varepsilon}|y(T)|^2 : y' + Ay + Fy = Bu, y(t) = x\Big\}. \tag{2.1}$$

For fixed ε, since the function $x \to \frac{1}{2\varepsilon}|x|^2$ is locally Lipschitz, we have that $\varphi_\varepsilon(t,\cdot)$ is locally Lipschitz for each $0 \le t \le T$. In fact, the Lipschitz constant on bounded sets is independent of t. For the proof, see [1].

Since $-A$ generates a compact semigroup, by usual compactness arguments, we can conclude that for fixed ε the infimum is attained in (2.1) and

$$\varphi_\varepsilon(t,x) \nearrow \varphi(t,x) \text{ as } \varepsilon \searrow 0 \text{ for each } (t,x) \in [0,T) \times H. \tag{2.2}$$

We also need the following lemma of independent interest:

Lemma 2.1. *Let A, F and B be defined as before. We consider $h, g, l_0, l_1 : H \to R$ such that h, g and l_1 are convex and C^1, and l_0 is merely continuous. If (u^*, y^*) is an optimal pair for the Bolza control problem,*

$$\inf\Big\{ \int_0^T (h(u) + g(y))dt + l_0(y(0)) + l_1(y(T)) : y' + Ay + Fy = Bu\Big\}, \tag{2.3}$$

then there exists $p \in C([0,T];H)$ satisfying

$$\begin{cases} p' - A^*p - (F'(y^*))^*p = \nabla g(y^*), \\ B^*p = \nabla h(u^*) \text{ a.e. on } (0,T) \\ p(T) = -\nabla l_1(y^*(T)) \text{ and} \\ \langle p(0), h\rangle \le \liminf_{\varepsilon\searrow 0}\frac{1}{\varepsilon}(l_0(y^*(0)+\varepsilon h) - l_0(y^*(0)))\ (\forall)\ h \in H. \end{cases}$$

Proof of Lemma 2.1. Let (u^*, y^*) be an optimal pair. If we denote $y_0 = y^*(0)$, we can see that (u^*, y^*) is also an optimal pair for the problem

$$\inf\Big\{ \int_0^T (h(u) + g(y))dt + l_1(y(T)) : y' + Ay + Fy = Bu,\ y(0) = y_0 \Big\}.$$

By [2], there exists $p \in C([0,T]; H)$ satisfying

$$\begin{cases} p' - A^*p - (F'(y^*))^*p = \nabla g(y^*) \\ B^*p = \nabla h(u^*) \text{ a.e. on } (0,T) \\ p(T) = -\nabla l_1(y^*(T)). \end{cases}$$

Now, we can fix u^* and take the initial data $y_0 + \varepsilon h$. We denote by $y_{y_0+\varepsilon h}$ the solution of the state system

$$y' + Ay + Fy = Bu^*, \quad y(0) = y_0 + \varepsilon h.$$

Since (u^*, y^*) is an optimal pair for the minimization problem (2.3), we see that

$$\begin{aligned} \int_0^T \frac{1}{\varepsilon}(g(y_{y_0+\varepsilon h}) - g(y^*))dt + \frac{1}{\varepsilon}(l_0(y_0 + \varepsilon h) - l_0(y_0)) \\ + \frac{1}{\varepsilon}(l_1(y_{y_0+\varepsilon h}(T)) - l_1(y^*(T))) \geq 0. \end{aligned} \tag{2.4}$$

Using compactness arguments (an infinite dimensional version of Arzelà–Ascoli), and the specific form of F, we obtain that

$$\frac{1}{\varepsilon}(y_{y_0+\varepsilon h} - y^*) \to z \text{ strongly in } C([0,T]; H),$$

where z is the solution of the variation system

$$z' + Az + F'(y^*)z = 0, \quad z(0) = h. \tag{2.5}$$

Since g and l_1 are C_1 on H, we can conclude from (2.4) that

$$\int_0^T \langle \nabla g(y^*), z\rangle dt + \operatorname{liminf}_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(l_0(y_0 + \varepsilon h) - l_0(y_0)) + \langle \nabla l_1(y^*(T)), z(T)\rangle \geq 0. \tag{2.6}$$

We now follow the usual computation

$$\begin{aligned} \langle z, p\rangle' &= \langle z', p\rangle + \langle z, p'\rangle \\ &= \langle -Az - F'(y^*)z, p\rangle + \langle z, A^*p + (F'(y^*))^*p + \nabla g(y^*)\rangle = \langle z, \nabla g(y^*)\rangle. \end{aligned}$$

By integration we obtain

$$\int_0^T \langle z, \nabla g(y^*)\rangle dt = \langle z(T), p(T)\rangle - \langle h, p(0)\rangle.$$

Substituting this in (2.6) we obtain

$$\langle z(T), p(T)\rangle - \langle h, p(0)\rangle + \langle \nabla l_1(y^*(T)), z(T)\rangle$$
$$+ \operatorname{liminf}_{\varepsilon\searrow 0}\frac{1}{\varepsilon}(l_0(y_0+\varepsilon h) - l_0(y_0)) \geq 0.$$

But $p(T) = -\nabla l_1(y^*(T))$, so we obtain

$$\langle p(0), h\rangle \leq \operatorname{liminf}_{\varepsilon\searrow 0}\frac{1}{\varepsilon}(l_0(y_0+\varepsilon h) - l_0(y_0)).$$

Since $h \in H$ was arbitrarily chosen, the proof of the lemma is complete.

We are now ready to state the main result of this section:

Theorem 2.1. *For each* $0 \leq t < T$, $\varphi(t, \cdot)$ *is locally Lipschitz. Moreover, if* (u^*, y^*) *is a pair for which the infimum is attained in* (1.5)*, then*

$$u^*(s) \in -B^*\partial\varphi(s, y^*(s)),\ \text{ a.e. } s \in [t, T).$$

Remark 2.1. In Theorem 2.1 we denoted by $\partial\varphi$ the generalized gradient in the sense of F. Clarke with respect to the state variable x. For the definition of the generalized gradient, see [2].

Proof of Theorem 2.1. From Carleman estimates in [4] we get that any solution of the adjoint equation

$$\begin{cases} p' + \triangle p + ap = 0 \text{ on } (t, T) \times \Omega \\ p(\tau, x) = 0 \text{ on } (t, T) \times \partial\Omega \end{cases}$$

satisfies the observability inequality:

$$\begin{aligned}\|p(t)\|^2_{L^2(\Omega)} &\leq C(T-t, \|a\|_{L^\infty(Q)}) \int_t^T \int_\omega |p(\tau, x)|^2 dx\ d\tau \\ &= C(T-t, \|a\|_{L^\infty(Q)}) \int_t^T \|B^*p(\tau)\|^2_{L^2(\Omega)} d\tau. \end{aligned} \tag{2.7}$$

Since $|f'(r)| \leq L$ we can obtain by Schauder's fixed-point theorem that the state system (1.2) is null controllable and

$$\varphi_\varepsilon(t, x) \leq \varphi(t, x) \leq \tfrac{1}{2}C(T-t, L)|x|^2. \tag{2.8}$$

For fixed $0 \leq t < T$, $\varepsilon > 0$ and $x \in H$, let us consider the minimization problem (2.1). Since the semigroup generated by $-A$ is compact, there exists an optimal pair (u^*, y^*) such that

$$\varphi_\varepsilon(t, x) = \int_t^T \frac{1}{2}|u^*|^2\ d\tau + \frac{1}{2\varepsilon}|y^*(T)|^2 \text{ and } y^{*\prime} + Ay^* + Fy^* = Bu^*,\ y(t) = x.$$

By the definition of φ_ε, (u^*, y^*) is also an optimal pair for the problem

$$\inf\Big\{ \int_t^T \frac{1}{2}|u|^2 d\tau + \frac{1}{2\varepsilon}|y(T)|^2 - \varphi_\varepsilon(t, y(t)) : y' + Ay + Fy = Bu \Big\}.$$

Considering $l_1(y) = \frac{1}{2\varepsilon}|y|^2$, and $l_0(y) = -\varphi_\varepsilon(t, y)$, by Lemma 2.1, we conclude there exists $p^{t,x} \in C([t, T]; H)$ such that

$$\begin{cases} (p^{t,x})' - A^* p^{t,x} - (F'(y^*))^* p^{t,x} = 0 \text{ on } [t, T] \\ p^{t,x}(T) = -\frac{1}{\varepsilon} y^*(T) \\ B^* p^{t,x} = u^* \text{ a.e on } [t, T] \end{cases} \tag{2.9}$$

and also

$$\langle p^{t,x}(t), h\rangle \leq \text{liminf}_{\lambda\searrow 0} \frac{-1}{\lambda}(\varphi_\varepsilon(t, x + \lambda h) - \varphi_\varepsilon(t, x)) \text{ for each } h \in H. \tag{2.10}$$

Replacing h by $-h$ in (2.10), we can see that whenever the limit

$$\lim_{\lambda\to 0} \frac{1}{\lambda}(\varphi_\varepsilon(t, x + \lambda h) - \varphi_\varepsilon(t, x)) = \frac{d}{d\lambda}(\varphi_\varepsilon(t, x + \lambda h))_{\lambda=0}$$

exists, then

$$\frac{d}{d\lambda}(\varphi_\varepsilon(t, x + \lambda h))_{\lambda=0} = \langle -p^{t,x}(t), h\rangle,$$

where $p^{t,x}$ satisfies (2.9). Using (2.7) for $p = p^{t,x}$ and $a = -f'(y^*)$, and then taking into account (2.8), we obtain

$$\begin{aligned} |p^{t,x}(t)|^2 &\leq C(T - t, L) \int_t^T |B^* p^{t,x}(\tau)|^2 d\tau = C(T - t, L) \int_t^T |u^*(\tau)|^2 d\tau \\ &\leq C(T - t, L) 2\varphi_\varepsilon(t, x) \leq C^2(T - t, L)|x|^2. \end{aligned}$$

So, we conclude that anytime $\frac{d}{d\lambda}(\varphi_\varepsilon(t, x + \lambda h))$ exists, then

$$\Big|\frac{d}{d\lambda}(\varphi_\varepsilon(t, x + \lambda h))\Big| \leq |p^{t,x+\lambda h}(t)||h| \leq C(T - t, L)|x + \lambda h||h|. \tag{2.11}$$

Now, for fixed ε and t, $\varphi_\varepsilon(t, \cdot)$ is locally Lipschitz. We fix $|x|, |y| \leq R$, so the function ψ defined by $\psi(\lambda) = \varphi_\varepsilon(t, x + \lambda(y - x))$ is Lipschitz on $[0, 1]$. This means that ψ is almost-everywhere differentiable and $\psi(1) - \psi(0) = \int_0^1 \psi'(\lambda) d\lambda$. So

$$\varphi_\varepsilon(t, y) - \varphi_\varepsilon(t, x) = \int_0^1 \frac{d}{d\lambda}\left(\varphi_\varepsilon(t, x + \lambda(y - x))\right) d\lambda.$$

Since $|x + \lambda(y - x)| \leq R$ for each $0 \leq \lambda \leq 1$, using (2.11) we can conclude that

$$|\varphi_\varepsilon(t, y) - \varphi_\varepsilon(t, x)| \leq C(T - t, L)R|y - x|, \text{ for } |x|, |y| \leq R.$$

So, on B_R, $\varphi_\varepsilon(t,\cdot)$ has the Lipschitz constant $C(T-t,L)R$, independent of ε. Passing to limit in (2.2) we conclude that $\varphi(t,\cdot)$ is Lipschitz on B_R with Lipschitz constant $C(T-t,L)R$.

We turn next to the existence of a feedback law for the minimum energy control. First we see that φ satisfies the dynamic programming principle

$$\varphi(t,x)=\inf\Big\{\int_t^s \frac{1}{2}|u|^2d\tau+\varphi(s,y(s));\ y\,'+Ay+Fy=Bu,\ y(t)=x\Big\} \tag{2.12}$$

for each $0\le t<s<T$ and each $x\in H$.

The proof of (2.12) is standard, so we skip it.

For fixed $0\le t<T$ we consider an optimal pair (u^*,y^*) on $[t,T]$; i.e.,

$$\varphi(t,x)=\int_t^T \frac{1}{2}|u^*|^2d\tau \text{ and } y^{*\,\prime}+Ay^*+Fy^*=Bu^*;\ y^*(t)=x, y^*(T)=0.$$

We know such a pair exists, since the semigroup generated by $-A$ is compact.

By usual dynamic programming arguments, we obtain that for a fixed $s\in(t,T)$, (u^*,y^*) is also an optimal pair on $[t,s]$ for the problem in (2.12). Since $\varphi(s,\cdot)$ is locally Lipschitz, by [2], we conclude that there exists $p^s\in C([t,s];H)$ such that

$$\begin{cases} (p^s)'-A^*p^s-(F'(y^*))^*p^s=0\\ B^*p^s=u^* \text{ a.e. on } [t,s]\\ p^s(s)\in-\partial\varphi(s,y^*(s)).\end{cases}$$

Here, by $\partial\varphi(s,\cdot)$ we denote the generalized gradient of $\varphi(s,\cdot)$. For $t<s_1\le s_2<T$, we consider the same arguments, obtaining the dual arcs $p^{s_1}\in C([t,s_1];H)$ and $p^{s_2}\in C([t,s_2];H)$. Since $B^*p^{s_1}=u^*=B^*p^{s_2}$ almost everywhere on $[t,s_1]$, and both p^{s_1} and p^{s_2} satisfy the adjoint system

$$p'-A^*p-(F'(y^*))^*p=0,$$

we conclude by observability inequality (2.7) (considered on subintervals of $[t,s_1]$ for $p=p^{s_1}-p^{s_2}$ and $a=-f'(y^*)$) that $p^{s_1}=p^{s_2}$ on $[t,s_1]$. Using this device for each $s\in[t,T)$, we obtain that there exists a unique $p\in C([t,T);H)$ such that

$$\begin{cases} p\,'-A^*p-(F'(y^*))^*p=0\\ B^*p=u^* \text{ a.e. on } [t,T)\end{cases} \tag{2.13}$$

and, furthermore, $p(s)\in-\partial\varphi(s,y^*(s))$ everywhere on (t,T). So we obtained the feedback law

$$u^*(s)\in\ -B^*\partial\varphi(s,y^*(s)) \text{ a.e. on } [t,T).$$

The proof of Theorem 2.1 is now complete.

Remark 2.2. By dynamic programming arguments we obtained also optimality conditions for the singular problem, namely (2.13). It remains to study the behaviour of p for $t \nearrow T$. In the linear case ($F = 0$), this is

$$\lim_{t \nearrow T} \langle y^*(t), p(t) \rangle = 0.$$

3. Viscosity approach

This section is devoted to the study of the Hamilton–Jacobi equation (1.6) subject to (1.7). In fact we want to characterize the value function φ as the viscosity solution (unique under extra assumptions) of (1.6) and (1.7).

Since we have to deal with quadratic growth solutions for the dynamic programming equation associated to a control problem, we will use the results in [6] (especially Section VII). First of all, for technical reasons, we rewrite (1.6) in the form

$$-\varphi_t + \langle Ax, \nabla\varphi \rangle + \langle Fx, \nabla\varphi \rangle + \tfrac{1}{2}|B^*\nabla\varphi|^2 = 0. \tag{3.1}$$

The definition of the viscosity solution for (3.1) is the usual definition for the unbounded linear case in [6], since A is a linear maximal monotone operator on H:

Definition 3.1. Let $u \in C([0,T)\times H)$. Then u is a subsolution (respectively, supersolution) of (3.1) if for every $\psi : [0,T) \times H \to R$, weakly sequentially lower semicontinuous such that $\nabla\psi$ and $A^*\nabla\psi$ are continuous and each g radial, nonincreasing and continuously differentiable on H and a local maximum (respectively, minimum) (t,z) of $u - \psi - g$ (respectively, $u + \psi + g$), we have

$$\begin{aligned}-\psi_t(t,z) + \langle z, A^*\nabla\psi(t,z)\rangle + \langle Fz, \nabla\psi(t,z) + \nabla g(t,z)\rangle \\ + \tfrac{1}{2}|B^*(\nabla\psi(t,z) + \nabla g(t,z))|^2 \le 0\end{aligned} \tag{3.2}$$

(respectively

$$\begin{aligned}\psi_t(t,z) - \langle z, A^*\nabla\psi(t,z)\rangle - \langle Fz, \nabla\psi(t,z) + \nabla g(t,z)\rangle \\ + \tfrac{1}{2}|B^*(\nabla\psi(t,z) + \nabla g(t,z))|^2 \ge 0).\end{aligned} \tag{3.3}$$

Since the particular A we use is self-adjoint and $-A$ generates a compact semigroup, we can choose $D := (I + A)^{-1}$ self-adjoint, positive and compact to satisfy

$$\langle (A^*D + D)x, x \rangle \ge |x|^2. \tag{3.4}$$

We denoted by D the operator B in [6] since we already denoted by B the control operator.

In order to interpret condition (1.7), we first split it in two parts:

$$\varphi(T,x) = +\infty \quad \text{for } x \neq 0, \quad \text{and} \tag{3.5}$$

$$\varphi(T,0) = 0. \tag{3.6}$$

We replace then the formal condition (3.5) by

$$\lim_{\substack{t\to T\\ y\to x}} \varphi(t,y) = +\infty \text{ for } x \neq 0. \tag{3.7}$$

Since we want the solutions of (3.1) subject to (3.6) and (3.7) restricted to $[0,s]$ (for $0 \leq s < T$) to fit in the framework of [6] (Section VII), we have to look for solutions u which satisfy the following uniform continuity conditions:

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there exists a modulus } m_\varepsilon \text{ such that} \\ |u(t,x) - u(t,y)| \leq m_\varepsilon(|x-y|) \text{ for } |x|,|y| \leq \frac{1}{\varepsilon},\ 0 \leq t \leq T-\varepsilon \end{cases} \tag{3.8}$$

and

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there exists a modulus } \rho_\varepsilon \text{ such that} \\ |u(\tau,x) - u(t,e^{-A(t-\tau)}x)| \leq \rho_\varepsilon(t-\tau) \text{ for } |x| \leq \frac{1}{\varepsilon},\ 0 \leq \tau \leq t \leq T-\varepsilon. \end{cases} \tag{3.9}$$

By a modulus, we mean here, as usual, a function $m : [0,\infty) \to [0,\infty)$ continuous, nonincreasing and subadditive such that $m(0) = 0$.

Now we can state the main result of this section:

Theorem 3.1. (i) *If there exists* $\psi \in C([0,T) \times H)$ *a positive viscosity supersolution of* (3.1)*satisfying the final condition* (3.7) *and also* (3.8) *and* (3.9)*, then the state system* (1.2) *is null controllable and* $\psi \geq \varphi$ *(where* φ *is defined in* (1.5)*).*

(ii) *If the state system* (1.2) *is null controllable and* $\varphi(t,\cdot)$ *is locally Lipschitz for each* $t < T$*, then* φ *is the unique positive viscosity solution of* (3.1) *satisfying* (3.7)*,* (3.8)*,* (3.9) *and*

$$\varphi(s,y(s)) \to 0 \text{ for } s \to T \tag{3.10}$$

whenever

$$\begin{cases} y' + Ay + Fy = Bu \text{ on } [t,T] \\ y(t) = x, \quad y(T) = 0 \end{cases}$$

for an $u \in L^2(t,T;H)$*.*

Remark 3.1. Condition (3.10) means that φ goes to zero along the admissible trajectories of the state system (1.3). We can regard it as a substitute for (3.6).

In order to prove Theorem 3.1 we need the following lemma, which is in fact a result in [6] (Section VII).

Lemma 3.1. *Let $g : H \to R$ uniformly continuously on bounded sets and $g \geq 0$.*

(i) *Then v defined by*

$$v(t,x) = \inf\left\{\int_t^T \frac{1}{2}|u|^2 d\tau + g(y(T)) :\ y' + Ay + Fy = Bu,\ y(t) = x\right\}$$

is positive, continuous on $[0,T] \times H$ and is also a solution of (3.1)*, subject to $v(T,x) = g(x)$ $(\forall)$ $x \in H$. Moreover, for each $R > 0$ there exist moduli m_R and ρ_R such that*

$$|v(t,x) - v(t,y)| \leq m_R(|x-y|) \text{ for all } 0 \leq t \leq T,\quad |x|,|y| \leq R \text{ and} \tag{3.11}$$

$$|v(\tau,x) - v(t, e^{-A(t-\tau)}x)| \leq \rho_R(t-\tau) \text{ for all } 0 \leq \tau \leq t \leq T,\quad |x| \leq R. \tag{3.12}$$

(ii) *If $u \in C([0,T] \times H)$ is a positive subsolution and $w \in C([0,T] \times H)$ is a positive supersolution of* (3.1)*, both satisfying* (3.11)*,* (3.12) *and $u(T,x) = w(T,x)$ $(\forall)$ $x \in H$, then $u \leq w$ on $[0,T] \times H$. Consequently, the value function v is the unique positive viscosity solution of* (3.1) *subject to $v(T,x) = g(x)$ $(\forall) x \in H$, under the extra assumptions* (3.11) *and* (3.12)*.*

Proof of Lemma 3.1. Since this is a result in [6](Section VII) we just need to verify the hypotheses. Because $u \to \frac{1}{2}|u|^2$ is convex coercive and $F : H \to H$ is globally Lipschitz, the assumptions concerning the Hamiltonian in equation (3.1) are fulfilled.

Regarding the comparison part (ii), we just have to see that conditions (3.11) and (3.12) together imply D-continuity (in fact weak sequential continuity, since D is compact) on $[0,T) \times H$. This is indeed true, since we have the inequality (see [5], page 259)

$$\|e^{-tA}x\|_D^2 + 2t|e^{-tA}x|^2 \leq e^{2t}\|x\|_D^2,$$

and consequently $e^{-tA} : (H, \|\cdot\|_D) \to (H, |\cdot|)$ is bounded for each $t > 0$. We consider here $\|x\|_D^2 = \langle Dx, x\rangle$. So u and w are D-continuous, and satisfy (3.11) and (3.12). We can conclude that $u \leq w$. The existence part (i) is stated in [6].

Proof of Theorem 3.1. (i) Let $\psi \geq 0$ be a viscosity supersolution of (3.1), satisfying (3.7), (3.8) and (3.9). Let $(t,x) \in [0,T) \times H$ and suppose $t < t_1 < t_2 < \cdots < t_n < \cdots < T$ satisfy $t_n \nearrow T$ for $n \nearrow \infty$. We define

$v : [0, t_1] \times H \to R$ by

$$v(\bar{t}, y) = \inf\Big\{ \int_{\bar{t}}^{t_1} \frac{1}{2}|u|^2 d\tau + \psi(t_1, y(t_1)) : y' + Ay + Fy = Bu,\ y(\bar{t}) = y \Big\}. \tag{3.13}$$

Since ψ satisfies (3.8) and (3.9), ψ satisfies (3.11) and (3.12) for $T = t_1$. By Lemma 3.1 (i), since $\psi(t_1, \cdot)$ is uniformly continuous on bounded sets and positive, v satisfies also (3.11) and (3.12), and v is a positive viscosity solution for (3.1) on $[0, t_1] \times H$. Since ψ is a positive viscosity supersolution and $\psi(t_1, x) = v(t_1, x)$ $(\forall)$ $x \in H$, by Lemma 3.1 (ii) we can conclude that $\psi \geq v$ on $[0, t_1] \times H$. This in particular means that $\psi(t, x) \geq v(t, x)$.

Since $-A$ generates a compact semigroup, there exists an optimal pair (u_1^*, y_1^*) on $[t, t_1]$ for the problem (3.13) considered when $(\bar{t}, y) = (t, x)$. So we have

$$\psi(t, x) \geq v(t, x) = \int_t^{t_1} \frac{1}{2}|u_1^*|^2 d\tau + \psi(t_1, y_1^*(t_1))$$

and

$$\begin{cases} y_1^{*\prime} + Ay_1^* + Fy_1^* = Bu_1^* \text{ on } [t, t_1] \\ y_1^*(t) = x. \end{cases}$$

We apply the same device on $[t_1, t_2]$ to find (u_2^*, y_2^*) such that

$$\psi(t_1, y_1^*(t_1)) \geq \int_{t_1}^{t_2} \frac{1}{2}|u_2^*|^2 d\tau + \psi(t_2, y_2^*(t_2))$$

and

$$\begin{cases} y_2^{*\prime} + Ay_2^* + Fy_2^* = Bu_2^* \text{ on } [t_1, t_2] \\ y_2^*(t_1) = y_1^*(t_1). \end{cases}$$

Using the same argument on $[t_2, t_3]$ and so on to $[t_{n-1}, t_n]$ and then matching the solutions we find a pair (u^*, y^*) defined on $[t, T)$ by $y^* = y_n^*$ on $[t_{n-1}, t_n]$ and $u^* = u_n^*$ almost everywhere on $[t_{n-1}, t_n]$, such that

$$\psi(t, x) \geq \int_t^{t_n} \frac{1}{2}|u^*|^2 d\tau + \psi(t_n, y^*(t_n)) \quad (\forall)\ n, \text{ and} \tag{3.14}$$

$$\begin{cases} y^{*\prime} + Ay^* + Fy^* = Bu^* \text{ on } [t, T) \\ y^*(t) = x. \end{cases} \tag{3.15}$$

From (3.14), since $\psi(t_n, y^*(t_n)) \geq 0$, we conclude that $u^* \in L^2(t, T; H)$, and consequently (3.15) implies that $y^* \in C([t, T]; H)$ (y^* is defined in T also). By condition (3.7) we have $y^*(T) = 0$. (Otherwise $\psi(t_n, y^*(t_n)) \to \infty$, which contradicts $\psi(t, x) \geq \psi(t_n, y^*(t_n))$.)

Using (3.14) we also obtain $\psi(t, x) \geq \int_t^T \frac{1}{2}|u^*|^2 d\tau$, so $\psi(t, x) \geq \varphi(t, x)$.

(ii) We know that φ satisfies the dynamic programming principle (2.12). Since $\varphi(s,\cdot)$ is locally Lipschitz and positive, we can use Lemma 3.1 (i) to conclude that φ is a positive viscosity solution on $[0,s)\times H$ and satisfies (3.11) and (3.12) on $[0,s]$ for each $0\leq s<T$. This implies that φ is a viscosity solution on $[0,T)\times H$. Also, choosing s and R such that $s>T-\varepsilon$ and $R>\frac{1}{\varepsilon}$, we can see that φ satisfies (3.8) and (3.9) on $[0,T)$. It is easy to prove that φ fulfills condition (3.10) since $0\leq\varphi(s,y(s))\leq\int_s^T\frac{1}{2}|u|^2d\tau$ whenever

$$y'+Ay+Fy=Bu,\quad y(t)=x,\ y(T)=0$$

for $u\in L^2(t,T;H)$. Passing to limit for $s\nearrow T$, we obtain (3.10). From Section 2 we know that $\varphi\geq\varphi_\varepsilon$ for each $\varepsilon>0$. Since

$$\lim_{\substack{s\to T\\ y\to x}}\varphi_\varepsilon(s,y)=\frac{1}{2\varepsilon}|x|^2$$

we easily obtain that φ must satisfy (3.7). So, the existence part is proved.

Regarding uniqueness, let ψ be a positive viscosity solution satisfying the assumptions. We already proved in (i) that $\psi\geq\varphi$. Now, using the same device in (i), for fixed $(t,x)\in[0,T)\times H$ and $s\in[t,T)$, we use Lemma 3.1 to get

$$\psi(t,x)=\inf\Big\{\int_t^s\frac{1}{2}|u|^2d\tau+\psi(s,y(s)):y'+Ay+Fy=Bu,\ y(t)=x\Big\}. \tag{3.16}$$

We fix (u,y) such that

$$y'+Ay+Fy=Bu,\quad y(t)=x,\ y(T)=0 \tag{3.17}$$

and $u\in L^2(t,T;H)$. From (3.16) we obtain that $\psi(t,x)\leq\int_t^s\frac{1}{2}|u|^2d\tau+\psi(s,y(s))$. Taking into account that $\psi(s,y(s))\to 0$ for $s\to T$ (since ψ satisfies (3.10)), we also have that

$$\psi(t,x)\leq\int_t^T\frac{1}{2}|u|^2d\tau$$

for each pair (u,y) which satisfies (3.17). So $\psi(t,x)\leq\varphi(t,x)$. We thus obtained that $\psi=\varphi$, and the proof of Theorem 3.1 is complete. □

Since we proved in Section 2 that φ is locally Lipschitz, we can use Theorem 3.1 to conclude that φ is either the minimal positive viscosity supersolution of (3.1) satisfying (3.7), (3.8) and (3.9) or the unique positive viscosity solution with the same properties and (3.10) as well. Since (3.10) is a substitute for (3.6), we can consider that the value function φ is the unique positive viscosity solution of (3.1) subject to the formal final condition (1.7), under the extra assumptions (3.8) and (3.9).

4. Final remarks

Remark 4.1. Although in the framework here considered the operators A, B and $F'(u)$ are self-adjoint, we wrote the optimality conditions in terms of A^*, B^* and $(F'(u))^*$, in order to be consistent with the usual notation for the adjoint system.

Remark 4.2. All statements and proofs in the previous sections can be extended to more general cases. Thus we can consider instead of $f \in C^1(R)$ a function f depending on the space variable also, $f : \Omega \times R \to R$ which is measurable in x, C^1 in r, satisfies $f(x, 0) = 0$ $(\forall) x \in \Omega$ and

$$\left|\frac{\partial}{\partial r} f(x,r)\right| \leq L \ (\forall) \ (x,r) \in \Omega \times R.$$

We still can write the state system in the form (1.2) if we denote by $F : H \to H$ the operator $F(u)(x) = f(x, u(x))$ for almost every $x \in \Omega$. Instead of the energy (1.4), we can consider the generalized energy

$$J(u) = \int_t^T (\tfrac{1}{2}|u|^2 + \tfrac{1}{2}|Cy|^2) d\tau \tag{4.1}$$

where C is a linear bounded operator from H to a different Hilbert space Y. There are choices of Y and C such that the energy (4.1) has a significant meaning.

In this case, as we pointed out, the state system has the abstract form (1.2), but the Hamilton–Jacobi equation becomes

$$\varphi_t - \langle (Ax + Fx), \nabla\varphi \rangle - \tfrac{1}{2}|B^*\nabla\varphi|^2 + \tfrac{1}{2}|Cx|^2 = 0, \tag{4.2}$$

with the same final condition (1.7). Since Lemma 3.1 is still valid in this case, the only thing that remains to check is whether the value function φ, defined in (1.5), replacing the energy by the generalized energy, is still locally Lipschitz. This is indeed true, since from [4] we have the stronger observability inequality

$$\begin{aligned} \|p(t)\|^2_{L^2(\Omega)} &\leq C_1(T - t, \|a\|_{L^\infty(Q)}) \\ &\quad \times \Big(\int_t^T \|B^* p(\tau)\|^2_{L^2(\Omega)} d\tau + \int_t^T \|Cy(\tau)\|^2_Y d\tau \Big) \end{aligned} \tag{4.3}$$

if p satisfies the adjoint system

$$\begin{cases} p' + \triangle p + ap = C^*Cy \quad \text{on } (t,T)\times\Omega \\ p(\tau,x) = 0 \quad \text{on } (t,T)\times\partial\Omega. \end{cases}$$

From here, we can deduce that $\varphi(t,x) \le \frac{1}{2}C_1(T-t)|x|^2$, and using again Lemma 2.1 and the observability inequality (4.3) we obtain that $\varphi(t,\cdot)$ has the Lipschitz constant $C_1(T-t,L)R$ on B_R.

Remark 4.3. The results presented are true for different kinds of homogenous boundary conditions of the type $y_\nu + \alpha y = 0$ on Σ, where by y_ν we denoted the normal derivative and $\alpha \ge 0$ is a constant. In this case we can define $A : D(A) \to A$ by $D(A) = \{u \in H^2(\Omega) : \ u_\nu + \alpha u = 0 \text{ on } \partial\Omega\}$ and $Au = -\triangle u$, and the state system still has the same semigroup form (1.2).

Remark 4.4. Since φ and φ_ε are weakly sequentially continuous on $[0,T)\times H$ and $\varphi_\varepsilon(t,x) \nearrow \varphi(t,x)$ as $\varepsilon \searrow 0$ for each $(t,x) \in [0,T)\times H$, we can use Dini's criterion to conclude that $\varphi_\varepsilon(t,x) \to \varphi(t,x)$ uniformly for (t,x) in the weak (sequentially) compact set $[0,s]\times B_R$, for each $0 \le s < T$.

Acknowledgment. The author express his warmest thanks to Professor V. Barbu for proposing this subject of study and for helpful suggestions.

REFERENCES

[1] V. Barbu and G. Da Prato, "Hamilton–Jacobi Equations in Hilbert Spaces," Research Notes in Mathematics, Pitman Advanced Publishing Program, Boston–London–Melbourne, 1983.

[2] V. Barbu, "Optimal Control of Variational Inequalities," Research Notes in Mathematics, Pitman Advanced Publishing Program, Boston–London–Melbourne, 1984.

[3] F. Clarke and R. Vinter, *The relationship between the maximum principle and dynamic programming*, SIAM J. Control Optim., 25 (1987), 1291–1311.

[4] A.V. Fursikov and O. Yu. Imanuvilov, "Controllability of Evolution Equations," Lecture Notes Series 34 (1996), RIM Seoul University, Korea.

[5] M.G. Crandall and P.L. Lions, *Hamilton Jacobi equations in infinite dimensions, Part IV, Unbounded linear terms*, J. Funct. Anal., 90 (1990), 237–283.

[6] M.G. Crandall and P.L. Lions, *Hamilton Jacobi equations in infinite dimensions, Part V, Unbounded linear terms and B-continuous solutions*, J. Funct. Anal., 97 (1991), 417–465.

[7] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer Verlag, 1983.

[8] M. Sirbu, *A Riccati equation approach to the null controllability of linear systems*, Comm. Appl. Anal., to appear.