A RICCATI EQUATION APPROACH TO THE NULL CONTROLLABILITY OF LINEAR SYSTEMS

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ABSTRACT: In this paper we study the null controlabillity of the pair (A, B) by means of the RICCATI equation associated (for now formaly) to the minimization problem:

$$\min \left\{ \int_0^T \|u(t)\|^2\, dt;\ u \in L^2(0,T;U),\ y' = Ay + Bu,\ y(0) = x,\ y(T) = 0 \right\}.$$

Some applications to linear parabolic systems are also considered.
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1. INTRODUCTION

Let H and U be two real HILBERT spaces, $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup, $B : U \to H$ or $B : U \to (D(A^*))'$ a linear operator and let us consider the problem

$$\min \left\{ \int_0^T \|u(t)\|^2\, dt;\ y' = Ay + Bu,\ y(0) = x,\ y(T) = 0 \right\}. \tag{P}$$

By means of the well-known arguments of dynamic programming, to the problem (P) we associate the RICCATI equation

$$P'(t) + A^* P(t) + P(t)A - P(t)BB^* P(t) = 0 \text{ on } [0, T) \tag{R}$$

subjected to the final condition

$$\langle P(T)x, x\rangle = +\infty \quad \text{for} \quad x \neq 0. \tag{F}$$

We recall that the pair (A, B) is *null controlable on* $[\,t, T\,]$ if for each $x \in X$ there exists $u \in L^2(t, T; U)$ such that

$$\begin{cases} y'(\tau) = Ay(\tau) + Bu(\tau) \quad \text{on} \quad [\,t, T\,] \\ y(t) = x, \quad y(T) = 0. \end{cases}$$

The solution y of the state equation is considered in the mild-like sense we will define precisely later.

The goal of this paper is to study the relationship between null controllability of the pair (A, B) on every interval $[\,t, T\,]$ and the existence and uniqueness properties of the problem (R) subjected to the condition (F).

The main idea is to exploit the uniqueness result, already known, for the RICCATI equation on smaller intervals $[\,0, T-\delta\,]$ and then to pass to the limit for δ tending to 0. We would like to emphasize that this idea works for the HAMILTON-JACOBI equations as well. The main problem there is in fact the lack of an satisfactory uniqueness result for viscosity solutions with quadratic growth (solutions corresponding to locally LIPSCHITZ final condition).

In section 2 we consider the case in which B is a bounded operator, while section 3 contains an example illustrating the abstract theory. In section 4 we analyze the case in which B is unbounded, while in section 5 we present an application to the boundary NEUMANN, or NEWTON controllability of the heat equation.

2. DISTRIBUTED CONTROL SYSTEMS

Let H and U be two real HILBERT spaces with H separable, let $A : D(A) \subset H \to H$ the infinitesimal generator of a C_0 semigroup $\{S(t),\ t \geq 0\}$ and $B \in L(U;H)$, where $L(U,H)$ is the space of all linear, bounded operators from the control space U to the state space H. We also denote by $L(H)$ the BANACH algebra of all linear, bounded operators from H to H and by $\Sigma(H)$ (respectivelly $\Sigma^+(H)$) the BANACH space of all symetric (respectivelly symetric and positive) operators acting in H.

Since B is continuous, for each $x \in H$ and $u \in L^2(t,T;U)$, there exists a unique mild solution of the CAUCHY problem for the state system, i.e.

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-s)A}Bu(s)\,ds$$

for each $\tau \in [\,t,T\,]$.

We assume familiarity with the basic concepts and results on evolution equations governed by generators of C_0 semigroups and we refer to PAZY [7] for details.

In BARBU-DA PRATO's book [1] the authors study the problem

$$P'(t) - A^*P(t) - P(t)A + P(t)BB^*P(t) = 0 \text{ on } (0,T] \tag{2.1}$$

$$P(0) = P_0. \tag{2.2}$$

In fact they prove an existence and uniqueness result for a more general problem. Like in [1], we denote by $C_S([\,0,T\,];\Sigma(H))$ the set of all mappings $S : [\,0,T\,] \to \Sigma(H)$ such that $S(\cdot)x$ is continuous on $[\,0,T\,]$ for each $x \in H$. By a *mild solution* of (2.1) subjected to (2.2) we mean a function $P \in C_S([\,0,T\,];\Sigma(H))$ satisfying

$$P(t)x = e^{tA^*}P_0e^{tA}x - \int_0^t e^{(t-s)A^*}P(s)BB^*P(s)e^{(t-s)A}x\,ds \tag{2.3}$$

for each $x \in H$. In [2] the authors show that a function P is a mild solution of (2.1) and (2.2) if and only if it is a *weak solution*, i.e., for each $x, y \in D(A)$ $\langle P(\cdot)x, y\rangle$ is differentiable on $[\,0,T\,]$ and verifies;

$$\frac{d}{dt}\langle P(t)x, y\rangle = \langle P(t)x, Ay\rangle + \langle P(t)Ax, y\rangle - \langle B^*P(t)x, B^*P(t)y\rangle. \tag{2.4}$$

By analogy, in what follows, we define the concept of mild, or weak solution for the backward CAUCHY problem associated to a RICCATI equation. Namely, we say that $P \in C_S([\,0,T\,];\Sigma(H))$ is a *mild solution* for the backward CAUCHY problem

$$P'(t) + A^*P(t) + P(t)A - P(t)BB^*P(t) = 0 \text{ on } [0,T) \tag{2.5}$$

$$P(T) = P_0 \tag{2.6}$$

if $\tilde{P}(t) = P(T-t)$ is a mild solution for the CAUCHY problem (2.1), (2.2). From Theorem 11 and Proposition 3 in [1] we obtain

Lemma 2.1. *Assume that $P_0 \in \Sigma^+(H)$. Then there exists a unique mild solution P in $C_S([0,T];\Sigma^+(H))$ of (2.5) and (2.6). Moreover:*

$$\langle P(t)x,x\rangle = \min_{u\in L^2(t,T;U)} \left\{ \int_t^T \|u(\tau)\|^2\, d\tau + \langle P_0 y(T), y(T)\rangle;\ y' = Ay + Bu,\ y(t) = x \right\}$$

and $u^*(\tau) = -B^* P(\tau) y^*(\tau)$ is the optimal feedback control on $[t,T]$.

Now let us replace the formal condition (F) by

$$\lim_{(s,y)\to(T,x)} \langle P(s)y,y\rangle = +\infty \quad \text{for each} \quad x \neq 0. \tag{2.7}$$

Let us observe now that if $P \in C_S([0,T];\Sigma^+(H))$ is the mild solution on $[0,T]$ of the problem (2.5), (2.6) then, for each $t \in (0,T)$, P is also a mild solution for (2.5) on $[0,t]$ with the final condition $P(t)$. From this remark, using the fact that "mild" means in fact "weak", we conclude that the concept of mild solution is rather local than global. Accordingly we introduce:

Definition 2.1. A function $P \in C_S([0,T);\Sigma^+(H))$ is called a *mild solution* for the problem (2.5), (2.7) if:

(i) for each $\delta \in (0,T)$, P is a mild solution on $[0,T-\delta]$
(ii) P satisfies (2.7).

Now we can state our main result.

Theorem 2.1.

(i) *If the problem (2.5), (2.7) has a mild solution then the pair (A,B) is null controllable.*
(ii) *If the pair (A,B) is null controllable then the problem (2.5), (2.7) has a unique mild solution with the property*

$$\lim_{t\to T}\langle P(t)y(t),y(t)\rangle = 0 \tag{$*$}$$

for every mild solution y of the state system $y' = Ay+Bu$, $y(t_0) = x$, $y(T) = 0$ and $u \in L^2(t_0,T;H)$.

Proof. (i) Let P be a mild solution of the RICCATI equation (2.5), (2.7). Let $x \in X$ and $t \in [0,T)$ and let us consider the closed-loop system

$$y'(\tau) = Ay(\tau) - BB^*P(\tau)y(\tau) \tag{2.8}$$

$$y(t) = x. \tag{2.9}$$

Since $P \in C_S([0,T);\Sigma^+(H))$, by the uniform boundedness principle, we conclude that, for each $\delta \in (0,T)$, $t \mapsto \|P(t)\|$ is bounded on $[0,T-\delta]$. So the closed-loop system (2.8), (2.9) has a unique mild solution on $[t,T)$ (it has a unique mild solution on every interval of the form $[t,T-\delta)$). By definition it follows that, for each $s \in (t,T)$, P is a mild solution of (2.5), on $[0,s]$. So, by Lemma 2.1, we have:

$$\langle P(t)x,x\rangle = \min_{u\in L^2(t,s;U)} \left\{ \int_t^s \|u(\tau)\|^2\, d\tau + \langle P(s)y(s), y(s)\rangle;\ y' = Ay + Bu,\ y(t) = x \right\}.$$

Moreover, the solution y^* of the closed-loop system (2.8), (2.9) is optimal on $[t,s]$. So, for $s \in (t,T)$ we have:

$$\langle P(t)x,x\rangle = \int_t^s \|u^*(\tau)\|^2 d\tau + \langle P(s)y^*(s), y^*(s)\rangle, \tag{2.10}$$

where

$$u^*(\tau) = -B^*P(\tau)y^*(\tau) \tag{2.11}$$

on $[\,t, T)$.

Since $P(s) \in \Sigma^+(H)$, from (2.10), it folllows that

$$\langle P(t)x, x\rangle \geq \int_t^s \|u^*(\tau)\|^2 d\tau$$

and so $u^* \in L^2(t, T; U)$. Accordingly $Bu^* \in L^2(t, T; H)$ and consequently y^* is defined on $[\,t, T\,]$ (also in T) and $y^* \in C([\,t, T\,]; H)$. By (2.10) and (2.7) we deduce that $y^*(T) = 0$ and so the pair (A, B) is null controllable on $[\,t, T\,]$.

(ii) (a) Uniqueness. Let P be a solution of (2.5), (2.7) which satisfies $(*)$ and let y^* be the solution of the closed-loop system (2.8), (2.9). Using the same arguments as before we conclude that $y^*(T) = 0$ and (2.10) holds. Using $(*)$ and passing to the limit in (2.10), we obtain

$$\langle P(t)x, x\rangle = \int_t^T \|u^*(\tau)\|^2 d\tau, \tag{2.12}$$

where

$$\begin{cases} y^{*\prime} = Ay^* + Bu^* \\ y^*(t) = x \ \text{ and } \ y^*(T) = 0. \end{cases}$$

Let $(y, u) \in C([\,t, T\,]; H) \times L^2(t, T; U)$ such that

$$\begin{cases} y' = Ay + Bu \\ y(t) = x \ \text{ and } \ y(T) = 0. \end{cases} \tag{2.13}$$

Using the same device as in (i) we get

$$\langle P(t)x, x\rangle = \min_{v\in L^2(t,s;U)} \left\{ \int_t^s \|v(\tau)\|^2 d\tau + \langle P(s)z(s), z(s)\rangle;\ z' = Az + Bv,\ z(t) = x \right\}.$$

So we have

$$\langle P(t)x, x\rangle \leq \int_t^s \|u(\tau)\|^2 d\tau + \langle P(s)y(s), y(s)\rangle. \tag{2.14}$$

Passing to the limit for $s \to T$ in (2.14) and using $(*)$, we deduce

$$\langle P(t)x, x\rangle \leq \int_t^T \|u(\tau)\|^2 d\tau \tag{2.15}$$

for every pair (y, u) which satisfies (2.13). From (2.12) and (2.15) it follows that

$$\langle P(t)x, x\rangle = \min_{u\in L^2(t,T;U)} \left\{ \int_t^T \|u(\tau)\|^2 d\tau;\ y' = Ay + Bu,\ y(t) = x,\ y(T) = 0 \right\}.$$

So the solution of (2.5), (2.7) which satisfies condition $(*)$ is unique.

(b) Existence. Assume that (A, B) is null controllable and let us denote

$$\varphi(t, x) = \min \left\{ \int_t^T \|u(\tau)\|^2 d\tau;\ y' = Ay + Bu,\ y(t) = x,\ y(T) = 0 \right\}$$

and

$$\varphi_\varepsilon(t, x) = \min \left\{ \int_t^T \|u(\tau)\|^2 d\tau + \frac{1}{\varepsilon}\|y(T)\|^2;\ y' = Ay + Bu,\ y(t) = x \right\}.$$

Since (A, B) is null controllable, by standard arguments, it follows that $\varphi_\varepsilon(t,x) \uparrow \varphi(t,x)$ for $\varepsilon \downarrow 0$. From Lemma 2.1 we deduce that $\varphi_\varepsilon(t,x) = \langle P_\varepsilon(t)x, x\rangle$, where P_ε is the mild solution of

$$P'_\varepsilon(t) + A^*P_\varepsilon + P_\varepsilon(t)A - P_\varepsilon(t)BB^*P_\varepsilon(t) = 0 \text{ on } [0,T) \tag{2.16}$$

$$P_\varepsilon(T) = \frac{1}{\varepsilon}I. \tag{2.17}$$

For each $t \in [0,T)$ we have $\langle P_\varepsilon(t)x, x\rangle \uparrow \varphi(t,x)$ for $\varepsilon \downarrow 0$. Since $P_\varepsilon(t) \in \Sigma^+(H)$ and $\varphi_\varepsilon(t,x) \le \varphi(t,x) \le C_t\|x\|^2$ (where $C_t > 0$ is the observability constant on $[t,T]$), by means of a diagonal process (using the separability of H and weak compactness arguments), we conclude that there exists at least one sequence $\varepsilon_n \downarrow 0$ such that $P_{\varepsilon_n}(t)x \rightharpoonup P(t)x$ for each $x \in H$. So

$$\varphi(t,x) = \langle P(t)x, x\rangle, \tag{2.18}$$

where $P(t) \in \Sigma^+(H)$ for $0 \le t < T$.

One may easily conclude that the value function satisfies the dynamic programming principle

$$\varphi(t,x) = \min\left\{\int_t^s \|u(\tau)\|^2 d\tau + \varphi(s, y(s));\ y' = Ay + Bu,\ y(t) = x\right\}. \tag{2.19}$$

Replacing (2.18) in (2.19) we get

$$\langle P(t)x, x\rangle = \min\left\{\int_t^s \|u(\tau)\|^2 d\tau + \langle P(s)y(s), y(s);\ y' = Ay + Bu,\ y(t) = x\right\}. \tag{2.20}$$

Using Lemma 2.1 we conclude that P is a mild solution of the Riccati equation (2.5) on $[0,s]$. Inasmuch as s is arbitrary, $P \in C_S([0,T); \Sigma^+(H))$ and satisfies (i) in Definition 2.1. We show next that P satisfies $(*)$. This means that, along the trajectories of the state system with $y(T) = 0$, φ tends to 0. But this is obvious because, for each $(t,x) \in [0,T) \times H$ and each admissible pair $(y,u) \in C([t,T];H) \times L^2(t,T;U)$ satisfying

$$\begin{cases} y' = Ay + Bu \\ y(t) = x \text{ and } y(T) = 0 \end{cases} \tag{2.21}$$

we have $0 \le \varphi(s, y(s)) \le \int_s^T \|u(\tau)\|^2 d\tau \to 0$ for $s \to T$.

Finally we have only to show that P satisfies (2.7). To this aim we will prove first that

$$\lim_{s\to T} \varphi(s,x) = +\infty \tag{2.22}$$

uniformly for x in convex,closed and bounded subsets in H which do not contain the origin. Obviously this implies

$$\lim_{(y,s)\to(x,T)} \varphi(s,y) = +\infty \tag{2.23}$$

for each $x \neq 0$, which in its turn implies (2.7). To prove (2.22) we will prove that

$$\lim_{s\to T} \varphi(s,x) = +\infty \tag{2.24}$$

for each $x \neq 0$. Indeed, let $x \neq 0$ and let us observe that $\varphi_\varepsilon(t,x) = \langle P_\varepsilon(t)x, x\rangle \to \frac{1}{\varepsilon}\|x\|^2$ for $t \to T$. Let $M \ge 0$ and choose $\varepsilon > 0$ such that $M < \frac{1}{\varepsilon}\|x\|^2$. For this ε there exists $\delta > 0$ such that $\varphi_\varepsilon(t,x) \ge M$ for $t \ge T - \delta$. Inasmuch as $\varphi(t,x) \ge \varphi_\varepsilon(t,x)$ we have $\varphi(t,x) \ge M$ for $t \ge T - \delta$ and this proves (2.24).

We postpone for the moment the proof of Theorem 2.1 in the favour of

Lemma 2.2. (Dini's generalized Lemma) *Let K be a compact topological space and $f_n : K \to R$ a sequence of lower semicontinuous functions. If $f_n \le f_{n+1}$ and, for each $x \in K$, $f_n(x) \uparrow +\infty$ as $n \to +\infty$ then $f_n \uparrow +\infty$ uniformly on K.*

Proof. Fix $M > 0$ and let us define $A_n = \{x \in K;\ f_n(x) \le M\}$. Since f_n is l.s.c., A_n is closed. Since $(f_n)_n$ is pointwise convergent to $+\infty$ we have $\cap_{n=1}^{+\infty} A_n = \emptyset$. Since K is compact, there exists $m \in \mathbb{N}$ such that $\cap_{n=1}^{m} A_n = \emptyset$. Recalling that $(f_n)_n$ is increasing we have $\cap_{n=1}^{m} A_n = A_m = \emptyset$. But this shows that $f_m(x) > M$ for each $x \in K$ and consequently $f_n(x) > M$ for each $n \ge m$ and each $x \in K$. $\square$

Proof of Theorem 2.1 (*continued*) For each $t < T$ and $x \in H$ we have $\varphi(t,x) = \langle P(t)x, x\rangle$ and so $\varphi(t,\cdot)$ is convex and continuous. Accordingly it is l.s.c. on H. Using this remark, (2.24) and the fact that $\varphi(t,x)$ is increasing in t for each fixed x, from Lemma 2.2, we obtain $\varphi(t,x) \uparrow +\infty$ uniformly on weakly compact subsets in H which do not contain the origin. This proves (2.22).

We actually showed that $\varphi(t,x) = \langle P(t)x, x\rangle$, where P is a mild solution of the problem (2.5), (2.7) which also satisfies the condition $(*)$. $\square$

From the proof we also see that $u(t) = -B^*P(t)y(t)$ is the optimal feedback control for the problem P. In view of *the dinamic programming principle* this is not surprising.

Remark 2.1. We may prove (2.23) directly (without using Lemma 2.2) but we preferred to show (2.22) which contains more information about the value function φ.

3. DISTRIBUTED CONTROLLABILITY OF THE HEAT EQUATION

Let us consider the distributed control heat equation

$$\begin{cases} y_t(t,\xi) = \Delta_\xi y(t,\xi) + \chi_\omega(\xi)u(t,\xi) & \text{in } Q = (0,T\,] \times \Omega \\ y(t,\xi) = 0 & \text{on } \Sigma = (0,T\,] \times \partial\Omega \\ y(0,\xi) = y_0(\xi) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary $\partial\Omega$, $\omega \Subset \Omega$ is a given subdomain and χ_ω is the characteristic function of ω. As usual, we rewrite the problem above in a Hilbert space frame as follows. Namely, take $H = L^2(\Omega)$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$, $Ay = \Delta_\xi y$ and $B : L^2(\Omega) \to L^2(\Omega)$, $Bu = \chi_\omega u$. So (3.1) may be equivalently rewritten as

$$\begin{cases} y' = Ay + Bu \\ y(0) = y_0. \end{cases} \tag{3.2}$$

We know that the pair (A,B) is null controlable on every interval $[\,t,T\,]$. This result, conjectured by Fursikov in 1992, has been proved first by Lebeau, Robbiano [6] and then by Fursikov, Imanuvilov [5]. So the abstract theory developed applies in this case. If P is the unique mild solution of the Riccati equation satisfying $(*)$, then, for each $y_0 \in L^2(\Omega)$, we have

$$\langle P(\tau)y_0, y_0\rangle_{L^2(\Omega)} = \min\left\{\int_\tau^T \|u(t)\|^2_{L^2(\Omega)}dt;\ y' = Ay + Bu,\ y(\tau) = y_0,\ y(T) = 0\right\}.$$

Moreover, $u(t) = -B^*P(t)y(t)$ is the optimal feedback control on $[\,\tau, T\,]$.

Remark 3.2. The abstract theory developed does not take into consideration some specific regularity properties which may hold in some particular cases. So, in the parabolic case considered the solution of the associated RICCATI equation is very smooth. Namely, as shown in REMARK 10, p. 70 in [1], in this case, since e^{tA} is analytic, $P \in C^\infty([0,T);\Sigma^+(H))$ and it is a classical solution.

4. ABSTRACT BOUNDARY CONTROL PROBLEMS

In this section, following [2], we will consider an abstract version of a parabolic equation with control on the boundary of NEUMANN or NEWTON type. Now the state equation is $y' = Ay + Bu$, where $A : D(A) \subset H \to H$ generates an analytic semigroup and $B = (\lambda_0 - A)D \in L(U;(D(A^*))')$, where $D \in L(U,H)$ and λ_0 is in $\rho(A)$. Actually we assume that

$$(HP)\begin{cases} \text{(i) } A \text{ generates an analytic semigroup of type } \omega_0 \text{ and } \lambda_0 \text{ is a real element in} \\ \rho(A) \text{ such that } \omega_0 < \lambda_0 \\ \text{(ii) there exists } \alpha \in (\frac{1}{2},1) \text{ such that } D \in L(U, D(A^\alpha)). \end{cases}$$

Here and thereafter $D(A^\alpha)$ is the domain of the fractional power $[\lambda_0 - A]^\alpha$ of the operator $\lambda_0 - A$. See PAZY[7].

Thus $B = [\lambda_0 - A]D = [\lambda_0 - A]^{1-\alpha}[\lambda_0 - A]^\alpha D$. If we denote $E = [\lambda_0 - A]^\alpha D$, from (HP), we have that $E \in L(U,H)$. The CAUCHY problem for the state equation may be rewritten as

$$\begin{cases} y'(t) = Ay(t) + [\lambda_0 - A]^{1-\alpha}Eu(t) \\ y(0) = y_0, \end{cases} \tag{4.1}$$

or in the mild form

$$y(t) = e^{tA}y_0 + \int_0^t [\lambda_0 - A]^{1-\alpha}e^{(t-s)A}Eu(s)\,ds.$$

In this case ($\alpha > \frac{1}{2}$), for every $y_0 \in H$ and $u \in L^2(0,T;U)$, we have $y \in C([0,T];H)$ (see [2]). So it makes sense to speak about null controllability of the state system (4.1). In this section we follow closely the arguments in section 2 and consequently we shall only outline the proofs and we shall insist only on those arguments which seem to be different. In section 2 we used the fact that null controllability is equivalent to a certain observability inequality. We shall see that for the boundary control problem here considered this equivalence also holds.

Lemma 4.3. *If $\alpha > \frac{1}{2}$, the state system is null controllable if and only if:*

$$\left\|e^{(T-t)A^*}\xi\right\|^2 \le C_t \int_t^T \left\|E^*[\lambda_0 - A^*]^{1-\alpha}e^{(T-s)A^*}\xi\right\|^2 ds \tag{4.2}$$

for every $\xi \in H$.

Proof. Let y be the mild solution of the state equation in (4.1) with the initial condition $y(t) = x$. We have

$$y(T) = e^{(T-t)A}x + \int_t^T [\lambda_0 - A]^{1-\alpha}e^{(T-s)A}Eu(s)\,ds.$$

Let us define the operator $L_t : L^2(t,T;U) \to H$ by

$$L_t(u) = \int_t^T [\lambda_0 - A]^{1-\alpha}e^{(T-s)A}Eu(s)\,ds.$$

Since $\alpha > \frac{1}{2}$, L_t is well-defined and continuous from $L^2(t,T;U)$ to H. Obviously the state system is null controllable on $[\,t,T\,]$ if and only if $R(e^{(T-t)A}) \subset R(L_t)$. Following [3] we have that this is equivalent to: there exists $C_t > 0$ such that:

$$e^{(T-t)A}(S_H(0,1)) \subset L_t(S_{L^2(t,T;U)}(0,\sqrt{C_t}). \tag{4.3}$$

Since L_t and $e^{(T-t)A}$ are linear continuous, there are weakly-weakly continuous and therefore $L_t(S_{L^2(t,T;U)}(0,\sqrt{C_t})$ and $e^{(T-t)A}(S_H(0,1))$ are closed and convex. Accordingly (4.3) is equivalent to

$$\sup_{\|x\|\le 1} \left\langle e^{(T-t)A}x,\xi\right\rangle \le \sup_{\|u\|_{L^2(t,T;U)}\le\sqrt{C_t}} \{\langle L_t(u),\xi\rangle\} \tag{4.4}$$

for every $\xi \in H$. A simple computational argument shows that (4.4) is equivalent to (4.2). □

Remark 4.3.

(i) Since $\alpha > \frac{1}{2}$, we have

$$\int_t^T \left\| E^*[\,\lambda_0 - A^*\,]^{1-\alpha} e^{(T-t)A^*}\xi \right\|^2 ds < +\infty$$

for every $\xi \in H$.

(ii) In the case of null controllability each $x \in H$ can be steered into the origin by means of a control $u \in L^2(t,T;U)$ such that $\|u\|_{L^2(t,T;U)} \le C_t\|x\|^2$. Accordingly, the value function $\varphi(t,x)$ (defined like in section 2) satisfies:

$$\varphi(t,x) \le C_t\|x\|^2 \tag{4.5}.$$

We can now present the main result in this section. We consider the Riccati equation

$$P'(t) + A^*P(t) + P(t)A - P(t)BB^*P(t) = 0 \text{ on } [0,T) \tag{4.6}$$

subjected to

$$\lim_{(y,s)\to(x,T)} \langle P(s)y,y\rangle = +\infty \tag{4.7}$$

for each $x \neq 0$. Following [2] we will define the solution to the backward Riccati equation with final value P_0. Let $T > 0$ and let us denote by $C_{S,\alpha}([\,0,T\,];\Sigma(H))$ the set of all $P \in C_S([\,0,T\,];\Sigma(H))$ satisfying:

(i) for each $x \in X$ and each $t \in [\,0,T)$, $P(t)x \in D([\,\lambda_0 - A^*\,]^{1-\alpha})$
(ii) $[\,\lambda_0 - A^*\,]^{1-\alpha}P \in C([\,0,T);L(H))$
(iii) $\lim_{t\to T}(T-t)^{1-\alpha}[\,\lambda_0 - A^*\,]^{1-\alpha}P(t)x = 0$ for each $x \in H$.

Let us denote by $V(t) = [\,\lambda_0 - A^*\,]^{1-\alpha}P(t)$.

Definition 4.2. A *mild solution* of (4.6) satisfying

$$P(T) = P_0 \tag{4.8}$$

is a mapping $P \in C_{S,\alpha}([\,0,T\,];\Sigma(H))$ which verifies

$$P(t)x = e^{(T-t)A^*}P_0e^{(T-t)A}x - \int_t^T e^{(s-t)A^*}V^*(s)EE^*V(s)e^{(s-t)A}x\,ds$$

for each $x \in H$. A *weak solution* of (4.6), (4.8) is a mapping $P \in C_{S,\alpha}([\,0,T\,];\Sigma(H))$ such that, for each $x, y \in D(A)$, $\langle P(\cdot)x, y\rangle$ is differentiable on $[\,0,T\,]$ and

$$\begin{cases} \dfrac{d}{dt}\langle P(t)x,y\rangle + \langle P(t)x,Ay\rangle + \langle P(t)Ax,y\rangle - \langle E^*V(t)x,E^*V(t)y\rangle = 0 \\ \\ P(T) = P_0. \end{cases}$$

Again weak solution and mild solution is one and the same concept. Combining the results in [2] we get:

Lemma 4.4. *Assume that $P_0 \in \Sigma^+(H)$. Then the problem (4.6), (4.8) has a unique mild solution $P \in C_{S,\alpha}([\,0,T\,];\Sigma^+(H))$. Moreover*

$$\langle P(t)x,x\rangle = \min\left\{\int_t^T \|u(\tau)\|^2 d\tau + \langle P_0 y(T), y(T)\rangle;\ y' = Ay + Bu,\ y(t) = x\right\}.$$

In addition, the closed-loop system

$$y(\tau) = e^{(\tau-t)A}x - \int_t^\tau [\,\lambda_0 - A\,]^{1-\alpha} e^{(\tau-s)A} EE^*V(s)y(s)\,ds$$

has a unique solution $y^* \in C([\,t,T\,];H)$ and $u^*(\tau) = -E^*V(\tau)y^*(\tau)$ is the optimal feedback control on $[\,t,T\,]$.

We can now define the solution of (4.6), (4.7).

Definition 4.3. By a *mild solution* of the problem (4.6), (4.7) we mean a function $P \in C_S([\,0,T);\Sigma^+(H))$ such that:

(i) for each $x \in X$ and each $t \in [\,0,T)$, $P(t)x \in D([\,\lambda_0 - A^*\,]^{1-\alpha})$
(ii) $[\,\lambda_0 - A^*\,]^{1-\alpha}P \in C([\,0,T);L(H))$ $(V(\cdot) \in C([\,0,T);L(H)))$
(iii) P is a mild solution of (4.6) on every interval $[\,0,T-\delta\,]$
(iv) P satisfies (4.7).

The main result in this section is

Theorem 4.2.

(i) *If the problem (4.6), (4.7) has a mild solution then the system (4.1) is null controllable.*
(ii) *If the state system is null controllable, then the problem (4.6), (4.7) has a unique mild solution satisfying $(*)$.*

Remark 4.4.

(1) The condition $(*)$ is kept from section 1.
(2) We dropped the condition (iii) from the definition of $C_{S,\alpha}([\,0,T\,];\Sigma(H))$.

Proof. (i) If P is a mild solution of (4.6), (4.7), we have $P \in C_{S,\alpha}([\,0,s\,];\Sigma^+(H))$ for each $s \in [\,0,T\,)$. Indeed, since $(s-t)^{1-\alpha} \to 0$ as $t \uparrow s$ and $[\,\lambda_0 - A^*\,]^{1-\alpha}P(\cdot) \in C([\,0,s\,];L(H))$, we have

$$\lim_{t\uparrow s}(s-t)^{1-\alpha}[\,\lambda_0 - A^*\,]^{1-\alpha}P(t)x = 0.$$

Thus P is the unique mild solution of (4.6) with T replaced by s (on $[\,0,s\,]$) with the final value $P(s)$. From now on we can follow the proof of (i) in Theorem 2.1. By Lemma 4.4 the closed-loop system has a unique mild solution $y^* \in C([\,t,T\,);H)$, etc. We obtain also that $u^* = -E^*V(\cdot)y^*(\cdot) \in L^2(t,T;U)$. So, since $\alpha > \frac{1}{2}$, we have $y^* \in C([\,t,T\,];H)$. As in section 1, we then get $y^*(T) = 0$.

(ii) The proof of the uniqueness is identical with its corresponding counterpart in section 1. The closed loop system has the form

$$y(\tau) = e^{(\tau-t)A}x - \int_t^\tau [\,\lambda_0 - A\,]^{1-\alpha} e^{(\tau-s)A} EE^* V(s)\,ds.$$

The existence part. Since all the trajectories are continuous then the dynamic programming principle holds (the analogue of (2.19), where φ is defined in a similar way). In view of (ii) in Remark 4.3 concerning Lemma 4.3, φ satisfies the same boundedness condition as that obtained in section 2 (see (4.5)). So $\varphi(t,x) = \langle P(t)x, x\rangle$. (It is easy to prove that, in these circumstances, we also have $\varphi_\varepsilon(t,x) \uparrow \varphi(t,x)$). From Lemma 4.4 we have that, for each $s \in [\,0,T)$, $P \in C_{S,\alpha}([\,0,s\,];\Sigma(H))$ and it is a mild solution on $[\,0,s\,]$ for (4.6). Accordingly, for each $s \in (0,T)$, $V(\cdot) = [\,\lambda_0 - A^*\,]^{1-\alpha}P(\cdot)$ is continuous from $[\,0,s)$ to $L(H)$. This clearly implies that $V \in C([\,0,T);L(H))$. The proof of $(*)$ and (4.7) follows exactly the same lines as those of the corresponding counterparts in section 2. $\square$

5. BOUNDARY CONTROLLABILITY OF THE HEAT EQUATION

The case of NEUMANN boundary conditions.
Let us consider the system

$$\begin{cases} y_t(t,\xi) = \Delta_\xi y(t,\xi) & \text{in } Q_T = (0,T\,] \times \Omega \\ y_\nu(t,\xi) = u(t,\xi) & \text{on } \Sigma_T = (0,T\,] \times \partial\Omega \\ y(0,\xi) = y_0(\xi) & \text{in } \Omega \end{cases} \tag{5.1}$$

and let us recall that this can be written in the form (4.1) by putting $H = L^2(\Omega)$, $U = L^2(\partial\Omega)$, $D(A) = \{x \in H^2(\Omega);\ x_\nu = 0 \text{ on } \partial\Omega\}$ and $Ax = \Delta x$. For $\lambda_0 = 1$ we have

$$D([\,\lambda_0 - A\,]^\alpha = \begin{cases} H^{2\alpha} & \text{if } 0 < \alpha < \frac{3}{4} \\ \{x \in H^{2\alpha}(\Omega);\ x_\nu = 0 \text{ on } \partial\Omega\} & \text{if } \frac{3}{4} < \alpha < 1. \end{cases}$$

We introduce the NEUMANN mapping $v \mapsto Nv = w$ from $L^2(\partial\Omega)$ to $L^2(\Omega)$, where $w = \Delta w$ in Ω and $w_\nu = v$ on $\partial\Omega$. It is well-known that $N \in L(L^2(\partial\Omega); H^{3/2}(\Omega))$ and thus we can consider any $\alpha \in (0,\frac{3}{4})$ (so $\alpha > \frac{1}{2}$), if we take $D = N$.

The result extends to NEWTON boundary conditions. Now the state system is:

$$\begin{cases} y_t(t,\xi) = \Delta_\xi y(t,\xi) & \text{in } Q_T = (0,T\,] \times \Omega \\ y_\nu(t,\xi) + \beta(y(t,\xi) - u(t,\xi)) = 0 & \text{on } \Sigma_T = (0,T\,] \times \partial\Omega \\ y(0,\xi) = y_0(\xi) & \text{in } \Omega, \end{cases} \tag{5.2}$$

where $\beta > 0$. Here $H = L^2(\Omega)$, $U = L^2(\partial\Omega)$, $D(A) = \{x \in H^2(\Omega);\ x_\nu + \beta x = 0 \text{ on } \partial\Omega\}$ and $Ax = \Delta x$. The details are left to the reader.

In [8] the author shows that, for every initial data y_0 in $L^2(\Omega)$, both (5.1) and (5.2) are null controllable on $[0,\tau]$ with controls in $L^2(0,\tau : L^2(\partial\Omega))$ for each $\tau > 0$. So we can apply the abstract theory developed to study the null controllability minimization problem.

Remark 5.5. The theory doesn't apply to the case of DIRICHLET boundary control for the heat equation. In this case the state system is of the form

$$\begin{cases} y' = Ay + [\,\lambda_0 - A\,]^{1-\alpha} Eu \\ y(0) = y_0, \end{cases}$$

but $\alpha \in (0, \frac{1}{4})$. For $\alpha < \frac{1}{2}$ the trajectories of the system are not continuous. Following [2], we can take $\beta \in (\frac{1}{2}-\alpha, \frac{1}{2}-\frac{\alpha}{2})$ in order to obtain that $[\lambda_0 - A]^{-\beta} y \in C([0,T]; H)$. Due to this fact we can consider β-null controllability, i.e. $[\lambda_0 - A]^{-\beta} y(T) = 0$. So, if the state system is β-null controllable, we can approximate the problem

$$\overline{\varphi}(t,x) = \min \left\{ \int_t^T \|u(\tau)\|^2 \, d\tau \right\} \tag{$\overline{P}$}$$

over all $u \in L^2(t,T;U)$ satisfying

$$\begin{cases} y' = Ay + [\lambda_0 - A]^{1-\alpha} E u \\ y(t) = x \\ [\lambda_0 - A]^{-\beta} y(T) = 0. \end{cases}$$

by the problems

$$\overline{\varphi}_\varepsilon(t,x) = \min \left\{ \int_t^T \|u(\tau)\|^2 \, d\tau + \frac{1}{\varepsilon} \|[\lambda_0 - A]^{-\beta} y(T)\|^2 \right\} \tag{$\overline{P_\varepsilon}$}$$

over all $u \in L^2(t,T;U)$ satisfying

$$\begin{cases} y' = Ay + [\lambda_0 - A]^{1-\alpha} E u \\ y(t) = x. \end{cases}$$

From [2] we have that $\overline{\varphi}_\varepsilon(t,x) = \langle P_\varepsilon(t)x, x\rangle$, where P_ε is the solution (see [2]) of the Riccati equation (4.6) subjected to $P_\varepsilon(T) = \frac{1}{\varepsilon}[\lambda_0 - A^*]^{-\beta}[\lambda_0 - A]^{-\beta}$. Following the arguments in Lemma 4.3 we obtain $\varphi(t,x) \le C_t \|x\|^2$. Using the same device in section 2 ($\varphi_\varepsilon(t,x) \uparrow \varphi(t,x)$) we get $\varphi(t,x) = \langle P(t)x, x\rangle$, where $P(t) \in \Sigma^+(H)$ for each $t \in [0,T)$. But, in order to use the dynamic programming principle and uniqueness results for the Riccati equation in [2], we should have that

$$P(t) = [\lambda_0 - A^*]^{-\beta} P_\beta(t) [\lambda_0 - A]^{-\beta},$$

where $P_\beta(t) \in \Sigma^+(H))$ for $0 \le t < T$. So we should have a sharper boundedness property of the form

$$\varphi(t,x) \le C_t \|[\lambda_0 - A]^{-\beta} x\|^2$$

which we don't know for now. The only thing we can say in these circumstances is that, in the case of null controllability (which holds for the heat equation subjected to Dirichlet boundary control) is that

$$\varphi(t,x) = \langle P(t)x, x\rangle$$

$P(t) \in \Sigma^+(H)$ for every $t \in [0,T)$ and $P_\varepsilon(t)x \rightharpoonup P(t)x$ as $\varepsilon \to 0$.

6. FINAL REMARKS

All statements and proofs from sections 2 and 4 can be extended straightforward to the case of a bounded observation $C \in L(H;Y)$ (Y Hilbert space). Here the Riccati equation is:

$$\begin{cases} P'(t) + A^* P(t) + P(t)A - P(t)BB^*P(t) + C^*C = 0 \\ \langle P(T)x, x\rangle = +\infty \quad \text{for} \quad x \ne 0 \end{cases} \tag{6.1}$$

subjected to the optimization problem

$$\min \left\{ \int_0^T (\|u(t)\|^2 + \|Cy(t)\|^2) \, dt : \; y' = Ay + Bu, \; y(0) = x, \; y(T) = 0 \right\}.$$

For the particular case $C = 0$ here considered we have

$$\varphi(t,x) = \|Q(t)^{-1/2}S(T-t)x\|^2$$

(see [4] p. 412) and so P has the exact representation

$$P(t) = \left(Q(t)^{-1/2}S(T-t)\right)^* \left(Q(t)^{-1/2}S(T-t)\right)$$

where

$$Q(t) = \int_t^T S(T-s)BB^*S^*(T-s)\,ds$$

is the controllability operator on $[\,t,T\,]$. So, in this case, the approximation of φ by φ_ε may seem to be unneccessary, except for proving (2.7) and (4.7). For the general case ($C \neq 0$) this approximation and the estimation

$$\varphi_\varepsilon(t,x) \leq \varphi(t,x) \leq C_t\|x\|^2$$

(which is also true for C bounded) is essential.

The uniqueness condition ($*$) is nothing else than a growth property. Another way of describing the appropriate solution P of (6.1) without using ($*$) is to say that P is the least solution. One can easily see from the proof that, if Q is another solution of (6.1), we have $P(t) \leq Q(t)$ on $[\,0,T)$.

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