

1. Prove that if  $P(x) = a + bx + cx^2$  is a quadratic polynomial with integer coefficients  $a, b, c$  and  $n$  is an integer, then

$$r \equiv s \pmod{n} \quad \text{implies} \quad P(r) \equiv P(s) \pmod{n} .$$

Bonus: prove that the same is true for every integer polynomial  $P$ .

2. Show that for every integer  $a$  not divisible by 11 there is another integer  $b$  with  $a \cdot b \equiv 1 \pmod{11}$ . Show also that this  $b$  is unique modulo 11, that is, show that if  $c$  is another integer with  $a \cdot c \equiv 1 \pmod{11}$  then  $b \equiv c \pmod{11}$ . (Hint: Bezout's Lemma is better than a case-by-case computation.)

3. (a) Find a solution in integers  $x, y$  to the equation  $13x + 21y = 4$ .

(b) Find another solution.

(c) Find another.

(d) Stop me from continuing this question *ad infinitum* by describing all the solution pairs  $(x, y)$ . (Hint: finding one solution is the hard part and you already did that; then if  $(x', y')$  were another solution, you'd have two equations to play with, one with  $x, y$  and one with  $x', y'$ . Subtract, rearrange terms, and see what you can conclude...)

4. Show that any two consecutive Fibonacci numbers are coprime. That is, for every  $n \geq 0$ ,  $\gcd(F_n, F_{n+1})=1$ , where

$$F_0 = 1, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1$$

5. Compute the gcd of  $A = 273413, B = 57575$

Challenge question: Ten islanders have a large pile of coconuts that they wish to share equally. One by one they count off the coconuts, to make sure all the piles are even, but at the end only nine coconuts remain in the last round; they would have nine coconuts left after dividing them as evenly as possible. So one member of the tribe is voted off the island and the remaining nine people start all over again to divide the coconuts. Unfortunately, by the end they discover they have 8 leftover coconuts, and cannot distribute those evenly amongst themselves. You can imagine how this goes: every time they vote someone off the island, the remaining members find they cannot evenly share the pile of coconuts because in the last round they have one coconut too few to share evenly. The problem is finally solved when the last two islanders find one leftover coconut, and one of those two throws the other one into the sea, keeping all the coconuts to himself, thus making the island Great Again.

How many coconuts were there in the first place?