I promised a proof of the following (rather than taking class time to do it):

**Theorem.** *The sum of the reciprocals of all primes diverges*

This theorem obviously implies that there is no end to the list of primes, but it actually says more. After all, there's no end to the list of powers of ten, either, but their reciprocals $1/10, 1/100, 1/1000, \ldots$ add up to $0.111\ldots = 1/9 < \infty$. So the theorem above says something like "the primes are not as sparse as the powers of 10". To get something even stronger than this theorem you'd need to prove a result akin to the Prime Number Theorem that I mentioned in class. (As we discussed, the PNT even allows us to estimate the rate of divergence of this series.)

The heart of this proof relies on the observation made in class the other day: if $\mathcal{P}$ is a set of prime numbers, then

$$\prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)^{-1} = \sum_{n\in\mathcal{N}} \frac{1}{n}$$

where $\mathcal{N}$ is the set of natural numbers whose prime divisors all lie in $\mathcal{P}$. You get this identity by recognizing each factor on the left side as the sum of the geometric series $1 + 1/p + 1/p^2 + \ldots$, together with the distributive law.

When $\mathcal{P}$ is the set of *all* prime numbers, then $\mathcal{N}$ is the set of all positive integers, so the right side of this identity is the harmonic series, which you know diverges. If instead $\mathcal{P}$ is a finite set, then $\mathcal{N}$ includes only some of the natural numbers, and in particular the series on the right side will converge (because the product on the left side is finite).

I can avoid all the expansion that comes from the distributive property if I take the logs of both sides of this identity. Thanks to the multiplicative property of logarithms, we deduce

$$\sum_{p\in\mathcal{P}} -\log\left(1 - \frac{1}{p}\right) = \log\left(\sum_{n\in\mathcal{N}} \frac{1}{n}\right)$$

Next I will use the Taylor series for the log function — actually just the Taylor polynomial and remainder theorem, which tells us that $\log(1 + x) \approx x$ when $x$ is small; more precisely

$$\log(1 + x) = x - R_x \qquad \text{where } 0 < R_x < x^2 \text{ if } |x| \leq 1/2$$

This means that for each prime $p$, the summand $-\log(1 - 1/p)$ differs from $1/p$ by a remainder which is smaller than $1/p^2$. Summing over all primes $p \in \mathcal{P}$ we find the left side of our last identity differs from

$$\sum_{a\in\mathcal{P}} \frac{1}{p}$$

by less than

$$\sum_{a\in\mathcal{P}} \frac{1}{p^2},$$

which is less than then "$p$-series" $\sum(1/n^2)$, which you know converges. (It happens to converge to $\pi^2/6 = 1.644934\ldots$.)

We conclude that the numbers

$$\sum_{a\in\mathcal{P}} \frac{1}{p} \qquad \text{and} \qquad \log\left(\sum_{n\in\mathcal{N}} \frac{1}{n}\right)$$

differ by less than 1.65 or so, no matter what set $\mathcal{P}$ of primes we consider. On the other hand, if $\mathcal{P}$ includes all the primes up to some bound $B$, say, then the set $\mathcal{N}$ will surely contain at least all the the integers less than $B$, and so as $B$ increases, the sum on the right will be larger than the partial sums of the harmonic series, which increase to infinity. It follows that the sum on the left is also increasing to infinity as we increase $B$, which is another way of saying that the sum of the reciprocals of all the primes is infinite.

By waving your hands with some sloppy estimates it's not hard to turn this into a fairly convincing argument that the sum of the reciprocals of all the primes less than $B$ is about $\log(\log(B))$, which is just the estimate that the Prime Number Theorem would provide. Actually getting all the details right is a lot harder!

You can check that $1/2 + 1/3 + 1/5 > 1$. The sum of all the primes up to 277 is about 2.00235 . To get the sum larger than 3, you have to add up all the primes up to 5195977. I don't know how long you have to wait until the sum exceeds 4; I think you have to go past $10^{18}$, maybe $10^{19}$, and that would take my machine a couple of centuries to accomplish. To get the sum past 5 would require a list of primes which has about as many elements as there are sub-atomic particles in the universe, so I reckon we'll never get them all written down. And yet: if you keep going long enough, the sum will diverge to infinity!