

Math 343K (Rusin) Exam 1, Mar 3, 2011. Please put your name on each sheet of paper you want me to grade. Please leave whitespace on which I can provide feedback.

- 1(a). Compute  $\gcd(629, 272)$  .  
1(b). Find integers  $x, y$  for which  $23x + 14y = 1$ .

2. The *Fibonacci numbers* are the numbers  $F_n$  in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

which satisfy the recurrence  $F_{n+2} = F_{n+1} + F_n$ .

Prove that for all integers  $n \geq 1$ ,  $F_{n+10} \equiv F_n \pmod{11}$ .

3. When we wish to work with the integers-modulo- $n$ , we can often use the same definitions we used for the integer, although the theorem involving these definitions may be different. For example, when  $a, b \in \mathbf{Z}_n$ , we will write  $a|b$  if there exists a  $c \in \mathbf{Z}_n$  with  $b = ac$ .

3a. Show that  $6|1$  in  $\mathbf{Z}_{11}$ . (Here and below, by “1” I really mean  $[1]_{11}$  and likewise for the other elements of  $\mathbf{Z}_n$ .)

3b. Does  $6|1$  in  $\mathbf{Z}_{10}$ ? Does  $6|2$  there? Explain.

3c. Show that if  $a$  and  $n$  are integers and  $d = \gcd(a, n)$ , then  $a|d$  in  $\mathbf{Z}_n$ . (Hint: Bezout’s Lemma)

4. Compute the orders of  $\alpha = (1234)(56)$ ,  $\beta = (154)(267)$  and  $\gamma = \alpha\beta$  in the symmetric group  $\text{Sym}(11)$ .

5. Suppose  $G$  is a group. Show that for all  $a, b \in G$  there is an  $x \in G$  with  $ax = b$ , and there is a  $y \in G$  with  $ya = b$ . Is  $x = y$ ?

I will also give a few points extra credit if you can explain what these results tell you about the group table for  $G$ .

6. Suppose  $G$  is a group containing two subgroups  $H$  and  $K$ , of orders  $m$  and  $n$ , respectively.

(a) Show that the intersection of  $H$  and  $K$  is also a subgroup of  $G$ .

Now suppose also that  $\gcd(m, n) = 1$

(b) Show that  $H \cap K = \{e\}$ .

(c) Show that if  $h, h' \in H$  and  $k, k' \in K$  and  $hk = h'k'$  then  $h = h'$  and  $k = k'$ .

Note: Answers to these questions may be found at

<http://www.math.utexas.edu/users/rusin/343K/ans1.pdf>