1. Here is the question that I already announced in class. Given any subset A of the real number line, we can creat new sets in two ways. First we can compute the complement $A^c = \mathbf{R} \backslash A$. Obviously $(A^c)^c = A$ so there's nothing new obtained by applying this operation twice in a row. Second we have discussed the *closure* of A, denoted $\bar{A}$, which is the union of A and its set A' of accumulation points. So $\bar{A}$ is generally a bit bigger than A, but since $(\bar{\bar{A}}) = \bar{A}$, again there's nothing new obtained by applying this operation twice in a row.

But what if we allow the use of both operations? For example if $A = (-\infty, 0)$ is the set of negative numbers, then with both operations we can obtain four distinct sets: $\bar{A} = (-\infty, 0]$, $A^c = [0, \infty)$, and $(\bar{A})^c = (0, \infty)$ in addition to A; the sets constructed as, say, $(\bar{A^c})$ or $((\bar{A^c}))^c$ will just be one of these four — for *this* starting set A.

With more interesting starting sets A, there can be more variety. I invite you to try, say, $A = (-\infty, 0) \cup \{1\}$ or $A = \mathbf{Q}$ (the set of rational numbers). So this prompts the challenge: what starting set A can you come up with that leads in this way to the largest number of distinct sets?

2. The subject of Real Analysis tries to prove powerful things about "typical functions" but Nature has a way of making that difficult: she keeps a lot of weird functions out of sight from our day-to-day experience in typical scientific applications. But they are out there and they are often related to weird subsets of the real line. We have already talked about the subset $\mathbf{Q}$ or $\mathbf{R}$, which is countable yet dense in $\mathbf{R}$, and we have used it to define some weird functions. Likewise we have discussed the Cantor Set and we can use it to create functions with unexpected properties.

One of the ways to create interesting functions from interesting sets E is to consider the *indicator function* 1_E of the set E, defined by

$$1_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

(It's also called the *characteristic function* of E and may be denoted as χ_E.)

(a) What can you say about $1_E \cdot 1_F$? What about $1_E + 1_F$?

(b) Where is 1_E continuous?

3. Suppose f is a function defined on the whole real line. Use the *definition* of continuity to prove that if f is continuous at a point a and $f(a) \neq 0$, then the function $g(x) = 1/f(x)$ is continuous at $x = a$.

4. section 5.2 #5

5. section 5.2 #14

Definition: if A is a set and $f : A \to A$ is any function, then a point $a \in A$ where $f(a) = a$ is called a *fixed point* of f.

6. Show that there can be continuous functions without fixed points.

7. Suppose $A = [0, 1]$ and $f : A \to A$ is a continuous function with $f(0) = 1$ and $f(1) = 0$. Show that f has a fixed point. (Hint: let $g(x) = f(x) - x$.)

Remark: it is a theorem that every continuous function on a closed and bounded interval has a fixed point.