M361K (Rusin) TEST 1 — March 2 2021
All questions count equally. Do your best. Remember the university Honor Code.

1. Prove the *De Morgan Law*: If A and B are subsets of $\mathbf{R}$ then

$$(\mathbf{R}\backslash A) \cap (\mathbf{R}\backslash B) = \mathbf{R}\backslash(A \cup B)$$

2. Suppose A and B are sets, and $f : A \to B$ is onto (i.e. f is a surjection). Show that if A is countable then B is also countable.

For **extra credit**, find some condition that can be inserted into this statement to make it true, and then *prove* that your statement is true: "If A and B are sets, and $f : A \to B$ is onto, and (*add some condition here!*), then if B is countable, then A is also countable.

3. An *idempotent* is a solution to the equation $X^2 = X$. There are many idempotent 2×2 matrices, for example $X = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. What are all the idempotents in $\mathbf{R}$?

You must justify every step of your solution by referring to one of the axioms of the real numbers (e.g. the associativity of addition), and identify the first step in your proof where you use an axiom of the real numbers that does not apply to the set of 2×2 matrices.

4. Prove the following: for all $a, b \in \mathbf{R}$ with $a < b$ there exists a rational number $c \in \mathbf{Q}$ for which $a < c\sqrt{2} < b$. (For example, if $a = 0$ and $b = 1$ we may use $c = 1/2$.)

5. Suppose S is a bounded set of positive real numbers, and let z be its least upper bound. Now let $T = \{2 - (1/s) \mid s \in S\}$. Show that T is also bounded and that its least upper bound is $2 - (1/z)$.

6. Use the definition of the convergence of a sequence to prove that the sequence $1, \frac{9}{5}, 2, \frac{25}{13}, \frac{9}{5}, \ldots, \frac{(n+2)^2}{n^2+4}, \ldots$ converges.