Math 362K (Rusin) — FINAL EXAM — May 8 2014.

Please write your name and EID:

Each problem is worth up to 25 points. Most of the points will be awarded based on your reasoning, so be sure to explain (in words) what you are doing.

1. Poker is a card game played with a standard 52-card deck. (Ask me if you don't know what that set of cards looks like.) In the game of poker, each player is given a randomly-selected set of 5 cards from this deck, called his "hand".

(a) What is the probability that a person's hand contains only red cards?

(b) How many hands are a "flush" (a set of 5 cards all of the same suit)?

Extra credit: what is the probability of being dealt a "full house" — a hand consisting of three cards of one number and two cards of another, e.g.

$$\{3\text{club}, 3\text{heart}, 3\text{diamond}, \text{King club}, \text{King spade}\}$$

2. Do you remember Joey with the messy sock drawer? It contains 2 white socks, 2 black socks, 2 red socks, and 16 other socks, none of which match. He needs a pair of matching socks. But now his strategy is: he will draw socks one at a time at random until he has a matching pair (and then he stops).

(a) What is the probability that he succeeds with the very first pair he draws?

(b) What is the probability that he succeeds precisely upon drawing his third sock?

3. In my experience, most UT students are good, hard-working students — all except the 25% of students who just guess on exams (instead of studying), and the 5% of students who cheat. The good students do well: 60% of them turn in above-average exam papers; only 10% of the guessers do so well, as well as 90% of the cheaters.

Sasha just turned in an above-average exam paper. What's the probability that Sasha is a cheater?

4. I don't play darts very well: when I play, the darts go all over the board, hitting the board (a disk with a 20cm radius) with a uniform distribution. Let R be the random variable that measures the distance (in cm) from the dart to the center of the board.

(a) Compute the cumulative distribution function for this random variable.

(b) Compute the pdf for R.

(c) Compute $E[R]$ and $\text{Var}(R)$.

5. Many clocks being shipped from the factory are tested to find out how long they last before they stop working, so now we know that this random variable has an exponential distribution with parameter $\lambda = \frac{1}{8}$. I just bought a used clock (of unknown age!) from Goodwill. What is the probability it will still be working 8 years after I buy it?

6. A large group of friends splits into sets of six to form Dining Clubs: every week they go out to dinner together and when the bill comes, they roll a die to determine who will pay the (entire) bill — Pat pays for everyone if they roll a "1", Chris pays if they roll a "2", etc. We can define random variables X_1 and X_2 on this large group of friends: X_1 counts the number of meals each person eats for free before the first time they have to pay the bill and similarly X_2 counts the number of free meals they get between the first and second time they pay the bill.

(a) The unluckiest person is the one for whom $X_1 = X_2 = 0$; what is the probability that this event occurs?

(b) Find the joint mass function of X_1 and X_2.

7. The joint density function of two random variables X and Y is given by

$$f(x,y) = \begin{cases} cxy & \text{if } x > 0,\ y > 0, \text{ and } x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) the value of c; (b) the pdf f_Y of the random variable Y;

8. I have just produced Dave's Big Book Of Mazes. Hundreds of mazes for kids to do. My publishers asked how long it takes to complete each one. I pointed out that time-to-completion is actually a random variable X and asked if they wanted the mean μ_X and standard deviation σ_X of this random variable. They said yes. I said I had no idea what the distribution of X was like. They got mad.

So I assembled thousands of volunteers and asked them each to complete 100 of the mazes and to report to me when they were finished. After 30 minutes, 10% of them were done. After another 30 minutes, an additional 20% of them were done. Then I halted the experiment because I now knew μ_X and σ_X! How did I do it?

(Hints: First of all, how does the Central Limit Theorem help here? State any assumptions you make about X.

Secondly, what information do you gain about the expected completion time for 100 mazes when I report what I said about the first 30 minutes? Do you think μ_X is more or less than 0.30 minutes? By a lot or by a little? You should be able to write a useful equation that has both "30min" and μ_X in it, among other things.

Finally, get a similar equation from the information about the next 30 minutes, and then solve these two equations.)